J. Fluid Mech. (1988), *vol.* 194, *pp.* 187–216
Printed in Great Britain

Nonlinear stability of a stratified shear flow in the regime with an unsteady critical layer

By S. M. CHURILOV AND I. G. SHUKHMAN
Siberian Institute of Terrestrial Magnetism, Ionosphere and Radio Wave Propagation (SibIZMIR), USSR Academy of Sciences, Irkutsk 33, P.O. Box 4, 664033 USSR

(Received 8 April 1987 and in revised form 3 December 1987)

In a previous paper (Churilov & Shukhman 1987*a*) we investigated the nonlinear development of disturbances to a weakly supercritical, stratified shear flow; we now report a continuation of that study. The degree of supercriticality of the flow is assumed not too small so that – unlike Paper 1 – the critical layer that appears in the region of resonance of the wave with the flow is an unsteady rather than viscous one. The evolution equation with cubic and quintic nonlinearity has been derived. The nonlinear term is non-local in time, i.e. depends on the entire preceding development of the disturbance. This equation has been used in the analysis of the evolution of an initially small disturbance. It is shown that where wave amplitude A is small enough ($A \ll \nu^{\frac{1}{2}}$, ν is the inverse of the Reynolds number), cubic nonlinearity dominates. In this case, as in Paper 1, the character of the evolution essentially depends on the sign of the quantity $(\eta-1)$, where η is the Prandtl number. However, independently of this sign the disturbance reaches – as it increases – the level $A \sim O(\nu^{\frac{1}{2}})$ and then quintic nonlinearity becomes dominant. At this stage an 'explosive' regime occurs and amplitude grows as $A \sim (t_0-t)^{-\frac{7}{4}}$. The results obtained, together with the findings of Paper 1, provide a full description of the development of small disturbances at a large (but finite) Reynolds number in different regimes which are determined by the degree of flow's supercriticality.

1. Introduction

This paper is a continuation of the study of the character of the evolution of weakly unstable disturbances of a small amplitude in a stratified shear flow that was initiated by Churilov & Shukhman (1987*a*, hereinafter referred to as Paper 1) and Brown, Rosen & Maslowe, (1981). With this end in view, as in Paper 1, we shall consider a flow $u = \tanh y$ near the stability boundary, i.e. when the Richardson number J is slightly less than $\frac{1}{4}(\frac{1}{4}-J \ll \frac{1}{4})$. An important role in the development of disturbances is known to be played by the critical layer (CL) which is produced in a region where the wave's phase velocity coincides with the flow velocity. We recall that according as which of the three scales viscous l_ν, unsteady l_t or nonlinear l_{N}, is greater, where

$$l_\nu = \nu^{\frac{1}{3}}, \quad l_t = \gamma, \quad l_{\mathrm{N}} = A^{\frac{2}{3}}, \tag{1.1}$$

the CL regime will be, respectively, a viscous, an unsteady or a nonlinear one. Here A is the disturbance amplitude, $\gamma = |A^{-1}\mathrm{d}A/\mathrm{d}t|$, and ν is the inverse of the Reynolds number; all quantities have dimensionless representations such that the typical density gradient and the shear flow velocity and width are equal to unity.

In Paper 1 the evolution of disturbances has been investigated in the regime of a viscous CL which is realized for sufficiently small amplitudes A and supercriticality

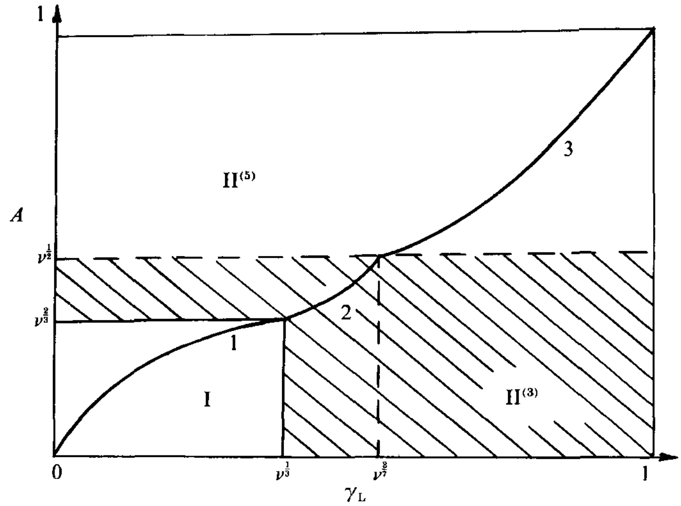

FIGURE 1. A diagram of the various regimes of the critical layer: I, the viscous CL region; II, the unsteady CL region (in II$^{(3)}$ the cubic nonlinearity is the principal one but in II$^{(5)}$ the quintic nonlinearity dominates). The thresholds of nonlinearity are: 1, $A = (\gamma_L \nu)^{\frac{1}{2}}$ for the viscous CL; 2, $A = (\gamma_L^7/\nu)^{\frac{1}{2}}$; and 3. $A = \gamma_L^{\frac{7}{4}}$ for the unsteady CL (Here $\eta - 1 = O(1)$ is assumed).

– with a relevant measure represented by the linear-theory growth rate $\gamma_L \sim (\frac{1}{4} - J)$. On the (A, γ_L) diagram (figure 1) this regime corresponds to Region I. The main results obtained in Paper 1 may be summarised as follows:

(i) The evolution of disturbances in the regime of a viscous CL is described by the Landau equation (Landau & Lifshitz 1959)

$$\frac{dA}{dt} = \gamma_L A + aA^3. \tag{1.2}$$

Terms of higher order (A^5, A^7, etc.) can be neglected (except for a very narrow region of values of Prandtl numbers near $\eta = 1$).

(ii) The Landau constant a is fully determined by the interaction of harmonics inside the CL and does not depend on the flow structure as a whole. It is directly proportional to the Reynolds number ($a = O(\nu^{-1})$) and changes its sign when $\eta = 1$.

(iii) Nonlinearity in the regime of a viscous CL is competitive (i.e. can affect substantially the development of a disturbance) throughout the entire range of variation of γ_L ($0 < \gamma_L < \nu^{\frac{1}{3}}$). When $\eta < 1$, it stabilizes the growth of disturbances at the level $A = A_0 = O((\gamma_L \nu)^{\frac{1}{2}})$ and when $\eta > 1$ it plays a destabilizing role: if $A > A_0$, the amplitude increases with time as $A \sim (t_0 - t)^{-\frac{1}{2}}$ until the CL regime is changed.

The regime of a viscous CL is realized only in a relatively small domain of the (A, γ_L)-diagram (figure 1). Let us define more precisely the boundaries of this domain and try to determine what the CL regime will be outside it, i.e. at sufficiently great supercriticality and (or) amplitude of the disturbance.

To begin with, we consider the region of very small amplitudes in which linear theory holds true. If supercriticality is small, we have Region I (a viscous CL). With larger supercriticality ($\gamma_L > \nu^{\frac{1}{3}}$), the unsteady scale l_t will exceed the viscous scale l_ν and the CL regime becomes unsteady. Let us now determine what the CL regime will be somewhat above Region I, i.e. with relatively small supercriticality ($\gamma_L < \nu^{\frac{1}{3}}$) but a sufficiently great amplitude. To do this, we must consider how the three scales (1.1) are competing. The nonlinear scale l_N will exceed the viscous one only when $\eta > 1$. In order to estimate the unsteady scale, we must remember that the disturbance can reach the upper boundary of a viscous CL only when $\eta > 1$, i.e. when nonlinearity contributes to growth rather than stops it. In this case, according to (1.2), $\gamma \sim aA^2 = O(A^2/\nu)$ and the unsteady scale has already become greater than the viscous one when $A > \nu^{\frac{2}{3}}$, i.e. at a much smaller amplitude than required for the regime of a nonlinear CL to set in. Consequently, above Region I the CL will also be an unsteady, rather than, as stated in Paper 1, nonlinear one. This more accurate definition does not alter the results obtained in Paper 1 for the regime of a viscous CL but only bounds the region of realization of this regime by a somewhat smaller amplitude than we believed earlier.

Thus, the region of a viscous CL is adjacent only to the region of an unsteady CL (see figure 1). Therefore, it would be logical to study how the nonlinear development of disturbances in the regime of an unsteady CL occurs, and this will be the subject of the present paper. Most of the paper is devoted to deriving the nonlinear evolution equation. Brown & Stewartson (1978) were the first to derive such an equation for the regime of an unsteady CL (with a somewhat different formulation of the problem). Unfortunately, the assumptions made in the derivation of it restricted the validity range to the very initial stage of development of the disturbance. We have been able to generalize the problem formulation and derive a nonlinear equation that is valid for describing the entire process of evolution up to amplitudes of order unity. A rather simple analysis (§3) shows that nonlinearity, as in the regime of a viscous CL, is fully determined by the interaction of harmonics inside the CL and does not depend on the flow structure as a whole. Consequently, all results obtained will be justified for a very broad class of flows and not only for the particular model that we have chosen to ease the calculations.

In complete form the nonlinear evolution equation is a series in A^2/λ^3, where λ is the CL scale, and may be written symbolically as

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \gamma_L A + \sum_{n=1}^{\infty} \frac{c_{2n} A^{2n+1}}{\lambda^{3n}}.$$

(In fact, in the regime of an unsteady CL the right-hand side is non-local in time, i.e. depends not only on the instantaneous value of amplitude but on the whole preceding evolution.) Until the CL becomes nonlinear, $\lambda \gg l_N = A^{\frac{2}{3}}$, i.e. A^2/λ^3 is the small parameter, and this allows us to neglect nonlinear terms of the highest order. In particular, in the regime of a viscous CL ($\lambda = \nu^{\frac{1}{3}}$) this immediately leads to equation (1.2), as long as the Prandtl number $\eta \neq 1$.

In the case of an unsteady CL ($\lambda = \gamma$), however, equations describing a solution inside the CL, possess a certain symmetry (see §3), by virtue of which, in the first approximation, $c_2 = 0$. (Note that in the regime of a viscous CL, this same symmetry arises when $\eta = 1$.) The symmetry is an approximate one and there are a number of factors that break it. Among them, the strongest factor (for $\eta \neq 1$) is associated with dissipative processes, namely the viscosity and heat conduction. With their inclusion, we get

$$c_2 \sim (\eta - 1)(l_\nu/l_t)^3 = (\eta - 1)\,\nu\gamma^{-3}.$$

Thus, in the regime of an unsteady CL, we must leave two nonlinear terms in the nonlinear evolution equation (with the remaining terms neglected):

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \gamma_L A + (\eta - 1)\,\nu b_2 \hat{\Gamma}_1^{-6} A^3 + b_4 \hat{\Gamma}_2^{-6} A^5, \tag{1.3}$$

where b_2 and b_4 are the coefficients, and $\hat{\Gamma}_{1,2}$ are certain non-local (integral) operators that correspond – in a rough approximation – to time integration; for estimation we may assume $\hat{\Gamma} = O(\gamma)$. It should be emphasized that in the nonlinear stage of development of disturbances it is not possible to identify γ with the linear increment γ_L because, as a rule, $\gamma \gg \gamma_L$.

Let us now define the 'spheres of influence' of each of the nonlinear terms in (1.3) as well as the boundaries of the unsteady CL region. Comparison of nonlinear terms shows that when $A < \nu^{\frac{1}{2}}$ the main contributor is the cubic nonlinearity (Region $\mathrm{II}^{(3)}$ in figure 1), while when $A > \nu^{\frac{1}{2}}$ it is the nonlinearity A^5 (Region $\mathrm{II}^{(5)}$ in figure 1). Another indicator of 'influential degree' for nonlinearity is its competitive ability, i.e. its ability to compete with the linear term $\gamma_L A$. It is clear that for a very small amplitude the nonlinear terms are not competitive and the disturbance grows exponentially with growth rate γ_L; in this range $\hat{\Gamma} = \gamma_L$. Cubic nonlinearity becomes competitive when

$$A \sim A_0^{(3)} = \gamma_L^{\frac{7}{2}} \nu^{-\frac{1}{2}}. \tag{1.4}$$

However, if $\gamma_L > \nu^{\frac{2}{7}}$, then the second nonlinear term becomes competitive still earlier, when

$$A \sim A_0^{(5)} = \gamma_L^{\frac{7}{4}}, \tag{1.5}$$

and it will precisely determine the nonlinear evolution of the disturbance. The boundary separating the regions of linear and nonlinear evolution (it is natural to call it the threshold of nonlinearity) is depicted in figure 1. Segment 1 of it lies in the region of a viscous CL, segment 2 corresponds to (1.4), and segment 3 corresponds to (1.5).

We already know the boundaries of the unsteady CL region 'from below', i.e. for Region I. The boundary 'from above' is reached when at least one of the inequalities $A \ll 1$, $\gamma \ll 1$, and $A^{\frac{2}{3}} \ll \gamma$ is violated. Failure to satisfy the first two of them means that the evolution regime has become strongly nonlinear and the perturbation theory underlying our analysis no longer is applicable. Violation of the last inequality would mean transition into the regime of a nonlinear CL. It is easy to see that such a transition is not realized. As a matter of fact, according to (1.3), in Region $\mathrm{II}^{(3)}$ in which cubic nonlinearity is dominant, $A \ll \nu^{\frac{1}{2}}$ and $\gamma \sim (\nu A^2)^{\frac{1}{7}} \gg A^{\frac{4}{7}} \gg A^{\frac{2}{3}}$. In Region $\mathrm{II}^{(5)}$, however, we have $\gamma \sim A^{\frac{4}{7}} \gg A^{\frac{2}{3}}$.

In summarizing the foregoing, let us formulate two major conclusions.

(i) Growth of small disturbances in a weekly supercritical, stratified shear flow (in contrast to unstratified flows, see e.g. Churilov & Shukhman 1987*b*) *does not lead* to the formation of a nonlinear critical layer. In other words, only two regimes of CL are realized, namely viscous and unsteady.

(ii) Within the framework of a weakly nonlinear ($A \ll 1$) theory the evolution of a disturbance is fully described (outside the region of a viscous CL) by (1.3). This means that (1.3) involves all required terms, each of which (in its respective region) has an essential influence upon the development of the disturbance, and the addition of other terms ($\sim A^7$, for example) merely provides small corrections.

In this paper we shall consider the nonlinear evolution of disturbances in Region $\mathrm{II}^{(3)}$ (cubic nonlinearity) and in Region $\mathrm{II}^{(5)}$ (quintic nonlinearity) and on combining the results with those obtained in Paper 1 we shall obtain an overall picture of perturbation development in a weakly supercritical stratified shear flow.

The paper is organized as follows. In §2 the problem is formulated and a brief account of the outer solution is given. In §3 the inner solution is derived as well as the evolution equation, and its analysis and solutions for different cases are considered in §4. A discussion is given in §5. Appendix A contains calculations supplementary to those in §3, and Appendix B† gives in explicit form the unwieldy expression for the kernel of the amplitude equation for the case of quintic nonlinearity.

2. Problem formulation and the outer solution

As in Paper 1, we use for calculations Drazin's model (Drazin 1958) in which the velocity u_0 along the x-axis and the density ρ_0 depend on the vertical coordinate y: $u_0 = \tanh y$, $\rho = \rho_{00} - y$. In Boussinesq's approximation the initial system of equations takes the form

$$\left.\begin{aligned} \frac{\partial}{\partial t}\Delta\psi - J\frac{\partial\rho}{\partial x} + \{\Delta\psi, \psi\} &= \nu\Delta^2\psi \equiv \eta\kappa\Delta^2\psi, \\ \frac{\partial\rho}{\partial t} + \{\rho, \psi\} &= \kappa\,\Delta\rho. \end{aligned}\right\} \tag{2.1}$$

Here we use the notation of Paper 1. The neutral stability curve in the inviscid limit is described by $J = J_0 = k^2(1-k^2)$ (Drazin 1958) and has a maximum $J = \frac{1}{4}$ at $k^2 = \frac{1}{2}$.‡

We consider a weakly supercritical flow at large Reynolds number ($\nu \ll 1, \eta = O(1)$). The unsteady CL regime occurs from above and to the right of the viscous CL region (see figure 1). In order to have linear theory as a starting point, we suppose at the beginning the supercriticality to be large enough ($\gamma_L > \nu^{\frac{1}{3}}$) and then extend the results obtained to the region above the viscous CL ($\gamma_L < \nu^{\frac{1}{3}}$, $A > \nu^{\frac{2}{3}}$).

Let us introduce the evolutionary time $\tau = \mu t$ and put

$$J = \frac{1}{4} + \mu J^{(1)}, \quad \nu^{\frac{1}{3}} \ll \mu \ll 1 \quad (J^{(1)} < 0). \tag{2.2}$$

Let ϵ be a small parameter characterizing the magnitude of perturbation: $A = \epsilon B$. Bearing in mind the derivation of the evolution equation, which is valid when the amplitude is smaller, of the same order as, and greater than the threshold of nonlinearity, we must do such scaling of the small parameters ϵ, κ, and μ that permits the linear term to be compared to the non-linear one. One needs to distinguish two cases: (*a*) cubic nonlinearity and (*b*) quintic nonlinearity. According to estimates reported in §1 (and as will be apparent from the subsequent calculations), one must suppose that

in the case of cubic nonlinearity $\mu = O(\kappa\epsilon^2/\mu^6)$, and
in the case of quintic nonlinearity $\mu = O(\epsilon^4/\mu^6)$

Now we formulate the boundary and the 'initial' conditions. It is our intention to investigate the time development of perturbations. However, the problem of nonlinear evolution of an arbitrary initial perturbation is though to be a very difficult one and therefore we restrict ourselves to the problem of an 'eigenmode' evolution.

Let an initial amplitude be very small such that linear theory is applicable. Then, in a time $t \sim \mu^{-1}$ an exponentially increasing eigenmode with $k^2 = \frac{1}{2}$ (the fundamental harmonic) becomes prominent and free oscillations with other k are damped. To investigate the subsequent perturbation evolution within the framework of linear

† Appendix B is available from the *Journal of Fluid Mechanics* Editorial Office.

‡ Strictly speaking, when dissipation factors are taken into account, the neutral stability curve is slightly deformed (a distortion of $O(\nu)$). However, this effect is unimportant for the processes concerned, with typical times less than ν^{-1} (see also Paper 1).

theory, one can proceed in two ways. On the one hand, the coordinate dependence of the eigenfunction can be taken as an initial condition (this is a 'restricted' initial-value problem); and on the other, one can assume that only this mode has existed in the past (up to $t = -\infty$). The result will evidently be the same.

In the nonlinear theory we cannot solve even the 'restricted' initial-value problem. Indeed, as the fundamental generates the forced oscillations in other harmonics $k_l = lk$ ($l = 0, 2, 3, \ldots$) we must set at $\tau = 0$, in a self-consistent manner, the spatial structure not only of the fundamental but also of all other harmonics. To avoid this difficulty we choose another way, namely we shall assume that the perturbation develops 'from nothing', i.e. that the amplitude is equal to zero at $\tau = -\infty$. As the boundary conditions we take a periodicity in x:

$$\psi = \ln(\cosh y) + \sum_{l=-\infty}^{\infty} \psi_l(\tau, y)\, \mathrm{e}^{\mathrm{i}lkx}, \quad \rho = \rho_{00} - y + \sum_{l=-\infty}^{\infty} \rho_l(\tau, y)\, \mathrm{e}^{\mathrm{i}lkx}; \tag{2.3a}$$

and a finite extent in y:

$$\psi_l \to 0, \quad \rho_l \to 0 \quad \text{as} \quad y \to \pm\infty. \tag{2.3b}$$

Here evidently $\psi_{-l} = \bar{\psi}_l$, $\rho_{-l} = \bar{\rho}_l$ and the overbar denotes complex conjugation; $k = 2^{-\frac{1}{2}}$.

The general scheme of calculations is the same as in Paper 1. For a cubic nonlinearity one needs only the fundamental ($l = \pm 1$), the second ($l = \pm 2$) and the zeroth ($l = 0$) harmonics. For quintic nonlinearity one needs the third ($l = \pm 3$) harmonics also. Equations (2.1) are solved separately for the outer region and for the inner one (CL), and the solutions are then matched.

The outer solution is in fact the same as in Paper 1. For this reason we present only a summary of results.

The fundamental of the outer solution is calculated in the form of an expansion:

$$\psi_1 = \epsilon\psi_1^{(1)} + \epsilon\mu\psi_1^{(2)} + \epsilon\kappa\psi_1^{(3)} + \ldots, \quad \rho_1 = \epsilon\rho_1^{(1)} + \epsilon\mu\rho_1^{(2)} + \epsilon\kappa\rho_1^{(3)} + \ldots. \tag{2.4}$$

The function $\psi_1^{(1)}$ is a solution of the Taylor–Goldstein equation

$$L_1 \psi_1^{(1)} = 0,$$

where

$$L_l = \frac{\partial^2}{\partial y^2} - [(lk)^2 - 2\,\mathrm{sech}^2\, y - \tfrac{1}{4}\,\mathrm{cotanh}^2\, y],$$

and is the neutral mode of the linear problem

$$\psi_1^{(1)} = B^{\pm}\phi_a(y), \quad \rho_1^{(1)} = -\psi_1^{(1)}\,\mathrm{cotanh}\, y; \quad \phi_a \equiv \sinh^{\frac{1}{2}}|y|\,\mathrm{sech}\, y \tag{2.5}$$

(the $\pm$ signs refer to $y \gtrless 0$, respectively; matching to the inner solution yields $B^- = -\mathrm{i}B^+$, which corresponds to the indentation rule from below). Then, perturbation theory is invoked to calculate $\psi_1^{(2)}$, $\psi_1^{(3)}$ and the required iterations ψ_0, ψ_2, and ψ_3.

These calculations rely on the process of 'harmonic generation', that is the fundamental being of $O(\epsilon)$ generates owing to the nonlinearity the zeroth and second harmonics of $O(\epsilon^2)$, an interaction with which will contribute to the fundamental and third harmonics of $O(\epsilon^3)$, etc. The thus-obtained part of the outer solution is a series in integer powers of ϵ where only even harmonics are involved at $O(\epsilon^{2n})$, whereas only odd harmonics appear at $O(\epsilon^{2n+1})$.

The other part of the perturbation comprises contributions each of which is a solution of the homogeneous equation for the corresponding (lth) harmonic

$$L_l \phi_l = 0 \tag{2.6}$$

which satisfies the boundary conditions (2.3*b*). Generally speaking, when $y > 0$ and $y < 0$, the solutions of (2.6) are different functions (we denote them as $\phi_l^{\pm}$) which are impossible to match together (if $l \neq \pm 1$) with the help of the indentation rule from below because none of the harmonics, except for the fundamental, is an eigenmode. Therefore such contributions appear in the outer solution *only in that order when it is required by matching with the inner solution.* In the problem of concern here such a contribution will be needed for the second harmonic. If, however, $l = \pm 1$, the relevant contribution has the form $\delta\psi_1 = d^{\pm}\phi_a$, and the inner solution determines it only up to a term proportional to $\psi_1^{(1)}$. In order to avoid such an ambiguity in the definition of wave amplitude we require orthogonality of $\delta\psi_1$ and $\psi_1^{(1)}$:

$$\int_{-\infty}^{\infty} \mathrm{d}y\, \overline{\psi_1^{(1)}}\, \delta\psi_1 = (\overline{B^+}d^+ + \overline{B^-}d^-)\int_0^{\infty} \mathrm{d}y\, \phi_a^2(y) = 0.$$

Hence, subject to the above-mentioned relation $B^- = -\mathrm{i}B^+$ we have $d^- = \mathrm{i}d^+$ (i.e. indentation of the singular point from above).

Thus, the harmonics are calculated in the form of expansions:

$$\psi_0 = \epsilon^2\psi_0^{(1)} + \epsilon^2\kappa\psi_0^{(2)} + \dots; \quad \psi_2 = \epsilon^2\psi_2^{(1)} + \epsilon^2\kappa\psi_2^{(2)} + c^{\pm}\phi_2^{\pm} + \dots; \quad \psi_3 = \epsilon^3\psi_3^{(1)} + \dots,$$

and similarly for ρ. When $y \to 0$ the asymptotic representation of $\phi_2^{\pm}$ in the general case has the form

$$\phi_2^{\pm} \sim |y|^{\frac{1}{2}}\,(\ln(\tfrac{1}{2}|y|) + q^{\pm}), \tag{2.7}$$

where $q^{\pm}$ are real numbers (because of the symmetry of the chosen flow model we have $q^+ = q^-$). In view of this, we obtain an asymptotic representation of the outer solution as $y \to 0$ ($y = \mu Y$, and $\rho_l = -2\partial p_l/\partial y$; for details of the calculations see Paper 1) which is required for matching with the inner solution:

the fundamental

$$\left.\begin{aligned} \psi_1 &= \epsilon\mu^{\frac{1}{2}}\left[B^{\pm}|Y|^{\frac{1}{2}} - \frac{\mathrm{i}}{2k}\frac{\partial B^{\pm}}{\partial\tau}\frac{|Y|^{\frac{1}{2}}}{Y}\right] + \epsilon\mu^{\frac{3}{2}}B^{\pm}|Y|^{\frac{1}{2}}[\tfrac{1}{2}J^{(1)}(2\ln 2 - \Lambda^2) \\ &\quad + a_1^{(2)\pm} \mp b_1^{(2)\pm}\Lambda] + \epsilon\kappa\mu^{-\frac{5}{2}}\frac{(5\eta+1)\mathrm{i}}{48k}B^{\pm}\frac{|Y|^{\frac{1}{2}}}{Y^3} + d^{\pm}|Y|^{\frac{1}{2}} + \dots; \\ p_1 &= \epsilon\mu^{\frac{1}{2}}\left[B^{\pm}|Y|^{\frac{1}{2}} - \frac{\mathrm{i}}{2k}\frac{\partial B^{\pm}}{\partial\tau}\frac{|Y|^{\frac{1}{2}}}{Y}\right] + \epsilon\mu^{\frac{3}{2}}B^{\pm}|Y|^{\frac{1}{2}}[\tfrac{1}{2}J^{(1)}(2\ln 2 - \Lambda^2 + 4\Lambda) \\ &\quad + a_1^{(2)\pm} \mp b_1^{(2)\pm}\Lambda] - \epsilon\kappa\mu^{-\frac{5}{2}}\frac{\mathrm{i}(\eta-7)}{48k}\frac{|Y|^{\frac{1}{2}}}{Y^3}B^{\pm} + d^{\pm}|Y|^{\frac{1}{2}} + \dots \end{aligned}\right\} \tag{2.8}$$

(here $\Lambda = \ln(\tfrac{1}{2}\mu|Y|)$);

the zeroth harmonic

$$\left.\begin{aligned} \frac{\partial\psi_0}{\partial\tau} &= \pm\frac{\epsilon^2\mu^{-1}}{2Y}\frac{\partial}{\partial\tau}|B^{\pm}|^2 \pm \epsilon^2\kappa\mu^{-4}\frac{3\eta-1}{8Y^3}|B^{\pm}|^2 + \dots, \\ \frac{\partial p_0}{\partial\tau} &= \pm\frac{\epsilon^2\mu^{-1}}{2Y}\frac{\partial}{\partial\tau}|B^{\pm}|^2 \pm \epsilon^2\kappa\mu^{-4}\frac{|B^{\pm}|^2}{4Y^3} + \dots; \end{aligned}\right\} \tag{2.9}$$

the second harmonic

$$\left.\begin{aligned} \psi_2 &= \pm\epsilon^2\mu^{-1}\frac{(B^{\pm})^2}{6Y} \mp \epsilon^2\kappa\mu^{-4}\frac{(91\eta+23)\,\mathrm{i}}{432kY^4}(B^{\pm})^2 + \mu^{\frac{1}{2}}c^{\pm}|Y|^{\frac{1}{2}}(\Lambda + q^{\pm}) + \dots, \\ p_2 &= \pm\epsilon^2\mu^{-1}\frac{(B^{\pm})^2}{6Y} \pm \epsilon^2\kappa\mu^{-4}\frac{(17\eta-131)\,\mathrm{i}}{432kY^4}(B^{\pm})^2 + \mu^{\frac{1}{2}}c^{\pm}|Y|^{\frac{1}{2}}(\Lambda + q^{\pm} - 2) + \dots; \end{aligned}\right\} \tag{2.10}$$

the third harmonic

$$\psi_3 = -\epsilon^3\mu^{-\frac{5}{2}}\frac{(B^\pm)^3}{18|Y|^{\frac{5}{2}}}+\dots, \quad p_3 = -\epsilon^3\mu^{-\frac{5}{2}}\frac{(B^\pm)^3}{18|Y|^{\frac{5}{2}}}+\dots; \tag{2.11}$$

as well as a modified solvability condition (MSC) (Paper 1)

$$b_1^{(2)+}+b_1^{(2)-} = -\frac{i\pi}{k}\frac{1}{B^+}\frac{\partial B^+}{\partial\tau} = -\frac{i\pi}{k}\frac{1}{B^-}\frac{\partial B^-}{\partial\tau} \tag{2.12}$$

which follows from the boundary conditions (2.3*b*) at $O(\epsilon\mu)$ of the expansion (2.4). Here $a_1^{(2)\pm}$, $b_1^{(2)\pm}$, $c^\pm$, and $d^\pm$ are unknown coefficients which are determined by matching with the inner solution.† The MSC (2.12) is the initial expression for the evolution equation; one of the objectives of this paper is to derive this equation.

3. The inner solution. The derivation of the evolution equation

The main purpose of this Section is to calculate $b_1^{(2)\pm}$ and thereby obtain the amplitude equation from the MSC (2.12). One can easily establish the asymptotic properties of that iteration of the inner solution in which the $b_1^{(2)\pm}$ will be determined. In view of (2.8) it is the fundamental harmonic iteration which has the term $m^\pm Y^{\frac{1}{2}}\ln(\frac{1}{2}\mu Y)$ in its asymptotic (as $Y\to\pm\infty$) expansion. Then the matching provides

$$B^+(b_1^{(2)+}+b_1^{(2)-}) = -(m^+-m^-). \tag{3.1}$$

In other words, we obtain the evolution equation in that order of perturbation where we first have $m^+ \neq m^-$.

For obtaining the inner solution we change to the variable $Y=\mu^{-1}y$ and the functions Ψ and Φ:

$$\psi = \tfrac{1}{2}\mu^2Y^2+\Psi, \quad \rho = \rho_{00}-\mu Y-2\mu^{-1}P_Y, \quad \Phi = \Psi - P.$$

The equations for Ψ and Φ obtained from (2.1) take into account a number of different factors: nonlinearity, supercriticality, and dissipative processes. For each of these factors we retain only the main term due to it and omit the other terms. This procedure yields

$$\left.\begin{aligned}\hat{\mathrm{N}}\Psi &= \tfrac{1}{2}\Phi_x-\mu^{-2}\{\Psi_Y,\Psi\}^*-2\mu J^{(1)}(\Psi_x-\Phi_x)+\eta\kappa\mu^{-3}\Psi_{YYY},\\ \hat{\mathrm{N}}\Phi &= -\mu^{-2}\{\Phi_Y,\Psi\}^*-2\mu J^{(1)}(\Psi_x-\Phi_x)+\kappa\mu^{-3}\Phi_{YYY}+(\eta-1)\kappa\mu^{-3}\Psi_{YYY},\end{aligned}\right\} \tag{3.2}$$

where $\hat{\mathrm{N}} = \left(\frac{\partial}{\partial\tau}+Y\frac{\partial}{\partial x}\right)\mathscr{D}-\frac{1}{2}\frac{\partial}{\partial x}, \quad \mathscr{D} = \frac{\partial}{\partial Y}, \quad \{a,b\}^* = a_xb_Y-a_Yb_x,$

and the subscripts denote differentiation with respect to the relevant variable. The solution is constructed by analogy with (2.2) in the form of a Fourier series in x. Each harmonic is represented as an expansion:

$$\Psi_1 = \epsilon\mu^{\frac{1}{2}}\Psi_1^{(1)}+\epsilon^3\mu^{-\frac{5}{2}}\Psi_1^{(2)}+\epsilon\mu^{\frac{3}{2}}\Psi_1^{(3)}+\epsilon\kappa\mu^{-\frac{5}{2}}\Psi_1^{(4)}+\epsilon^3\kappa\mu^{-\frac{11}{2}}\Psi_1^{(5)}+\epsilon^5\mu^{-\frac{11}{2}}\Psi_1^{(6)}+\dots,$$

$$\Psi_0 = \epsilon^2\mu^{-1}\Psi_0^{(1)}+\epsilon^2\kappa\mu^{-4}\Psi_0^{(2)}+\epsilon^4\mu^{-4}\Psi_0^{(3)}+\dots,$$

$$\Psi_2 = \epsilon^2\mu^{-1}\Psi_2^{(1)}+\epsilon^2\kappa\mu^{-4}\Psi_2^{(2)}+\epsilon^4\mu^{-4}\Psi_2^{(3)}+\dots,$$

$$\Psi_3 = \epsilon^3\mu^{-\frac{5}{2}}\Psi_3^{(1)}+\dots,$$

and similarly for Φ.

† Note that the coefficients $a_1^{(2)\pm}$ and $b_1^{(2)\pm}$ have a known order ($O(1)$) while the order of the coefficients of 'free' solutions $c^\pm$ and $d^\pm$ is still to be determined by matching.

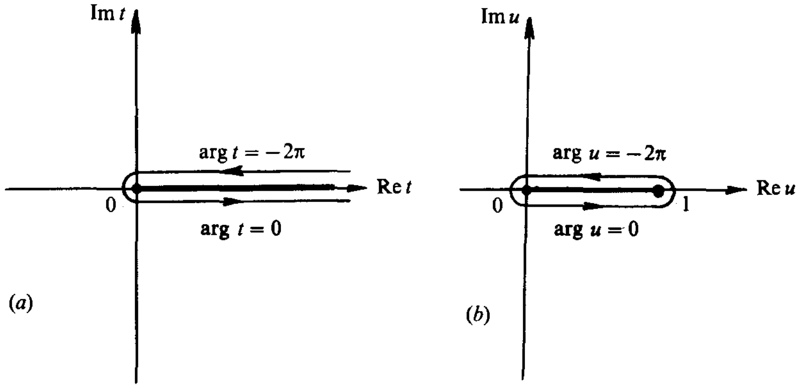

FIGURE 2. (*a*) contour C; (*b*) contour L.

The subsequent calculations are based on the solution of equations of the form $\hat{\mathrm{N}}F = R$ which is considered in detail in Appendix A. Therefore, in appropriate passages we shall give the results only, without going into details.

3.1. $O(\epsilon\mu^{\frac{1}{2}})$ *of the fundamental*

On writing

$$\hat{\mathrm{N}}_l = \left(\frac{\partial}{\partial\tau} + \mathrm{i}klY\right)\mathscr{D} - \tfrac{1}{2}\mathrm{i}kl,$$

we have

$$\hat{\mathrm{N}}_1\,\Psi_1^{(1)} - \tfrac{1}{2}\mathrm{i}k\Phi_1^{(1)} = 0, \quad \hat{\mathrm{N}}_1\,\Phi_1^{(1)} = 0.$$

A solution of these equations which matches to (2.8) is

$$\Psi_1^{(1)} \equiv W(\tau, Y) = \frac{\mathrm{e}^{3\mathrm{i}\pi/4}}{4(\pi k)^{\frac{1}{2}}}\int_C \mathrm{d}t\, t^{-\frac{3}{2}} B(\tau - t)\,\mathrm{e}^{-\mathrm{i}kYt}, \quad \Phi_1^{(1)} = 0. \tag{3.3}$$

Contour C is shown in figure 2(*a*). The function $W(\tau, Y)$ has a single asymptotic representation in the lower half-plane $(-\pi \leqslant \arg Y \leqslant \pi)$ of a complex Y as $|Y| \rightarrow \infty$:

$$W(\tau, Y) = B(\tau)\,Y^{\frac{1}{2}} + O(Y^{-\frac{1}{2}}), \tag{3.4}$$

from which it is evident that $B(\tau)$ is a wave amplitude (see (2.8) and (2.5)) and $B(\tau) = B^+ = \mathrm{i}B^-$. The evolution equation will be obtained just for $B(\tau)$. For the moment, however, it is an arbitrary function that satisfies the requirement $B(\tau) \rightarrow 0$ as $\tau \rightarrow -\infty$.

3.2. $O(\epsilon^2\mu^{-1})$ *of the zeroth harmonic*

Equations (3.2) give

$$\frac{\partial}{\partial\tau}\Psi_0^{(1)\prime} = \mathrm{i}k(W\overline{W}' - \text{c.c.})', \quad \frac{\partial}{\partial\tau}\Phi_0^{(1)\prime} = 0.$$

From here on, the prime denotes a derivative with respect to Y. The solution which matches to (2.9) is

$$\Psi_0^{(1)} = 2|W'|^2, \quad \Phi_0^{(1)} = 0.$$

3.3. $O(\epsilon^2\mu^{-1})$ *of the second harmonic*

$$\hat{\mathrm{N}}_2\,\Psi_2^{(1)} = \mathrm{i}k\Phi_2^{(1)} + \mathrm{i}k(W''W - W'^2), \quad \hat{\mathrm{N}}_2\,\Phi_2^{(1)} = 0.$$

Reasoning along similar lines as in Appendix A, it is easy to see that because the right-hand sides do not involve $\overline{W}$ and its derivatives, the solution has a single asymptotic representation (from here onward, as $|Y| \to \infty$ in the lower half-plane of a complex Y) which, subject to (3.4), is

$$\Psi_2^{(1)} \sim \frac{B^2}{6Y} + (m_2^{(1)} \ln(\tfrac{1}{2}\mu Y) + n_2^{(1)})\, Y^{\frac{1}{2}} + \ldots, \quad \Phi_2^{(1)} = 2m_2^{(1)} Y^{\frac{1}{2}} + \ldots.$$

Comparison with (2.10) indicates ($P = \Psi - \Phi$) that matching requires that $c^{\pm} = \epsilon^2 \mu^{-\frac{3}{2}} c_1^{\pm}$. Hence we obtain

$$c_1^+ = m_2^{(1)}, \quad c_1^- = -\mathrm{i} m_2^{(1)}, \quad c_1^+ q^+ = n_2^{(1)}, \quad c_1^- q^- = -\mathrm{i} n_2^{(1)} - \pi m_2^{(1)},$$

which yields the equation $m_2^{(1)} q^- = (q^+ - \mathrm{i}\pi)\, m_2^{(1)}$, and because $q^{\pm}$ is real its unique solution will be $m_2^{(1)} = 0$ (and $c_1^{\pm} = n_2^{(1)} = 0$).

Thus, we have $\Phi_2^{(1)} = 0$. Using (3.3) as well as formulae reported in Appendix A we obtain

$$\Psi_2^{(1)} = -\frac{\mathrm{i}k}{32\pi} \int_0^{\infty} \mathrm{d}s_1 \int_C \mathrm{d}s_2 \int_C \mathrm{d}s_3\, B(\tau - s_1 - s_2)\, B(\tau - s_1 - s_3)(s_2 s_3)^{-\frac{3}{2}} (s_2 + s_3)^{\frac{1}{2}} \times (2s_1 + s_2 + s_3)^{-\frac{3}{2}} (s_2 - s_3)^2\, \mathrm{e}^{-\mathrm{i}kY(2s_1 + s_2 + s_3)}.$$

Upon changing to the variables $t_1 = s_1 + s_2$ and $t_2 = s_1 + s_3$ and integrating over s_1, we have

$$\Psi_2^{(1)} = \frac{\mathrm{i}k}{4\pi} \int_0^{\infty} \mathrm{d}t_1 \int_C \mathrm{d}t_2\, B(\tau - t_1)\, B(\tau - t_2)\, t_1^{\frac{1}{4}} t_2^{-\frac{1}{4}} (t_1 + t_2)^{-\frac{3}{2}} (t_1 - t_2)^{\frac{1}{2}} \times F\left(-\frac{1}{2}, -\frac{1}{4}; \frac{3}{4}; \frac{t_2^2}{t_1^2}\right) \mathrm{e}^{-\mathrm{i}kY(t_1 + t_2)}\, \theta(t_1 - t_2), \tag{3.5}$$

where $F(a, b; c, z)$ is a hypergeometric function (Abramowitz & Stegun 1964),

$$\theta(z) = \begin{cases} 1, & z > 0, \\ 0, & z < 0. \end{cases}$$

3.4. $O(\epsilon^3 \mu^{-\frac{5}{2}})$ *of the fundamental*

The system (3.2) of this order yields

$$\hat{\mathrm{N}}_1 \Psi_1^{(2)} = \tfrac{1}{2}\mathrm{i}k\Phi_1^{(2)} + \mathrm{i}k(\Psi_0^{(1)\prime\prime} W - \Psi_0^{(1)\prime} W' + 2\Psi_2^{(1)} \overline{W}'' - \Psi_2^{(1)\prime} \overline{W}' - \Psi_2^{(1)\prime\prime} \overline{W})$$
$$\equiv \tfrac{1}{2}\mathrm{i}k\Phi_1^{(2)} + \mathrm{i}kR_1^{(2)},$$
$$\hat{\mathrm{N}}_1 \Phi_1^{(2)} = 0.$$

The function $R_1^{(2)}$ involves $\overline{W}$ and, therefore, does not have a single asymptotic representation. Nevertheless, $\Phi_1^{(2)} = 0$ because the expansion of ϕ_a as $y \to 0$ does not involve the term $\sim |y|^{\frac{1}{2}} \ln(\frac{1}{2}|y|)$. The equation for $\Psi_1^{(2)}$ has a solution (contour L is shown in figure 2*b*)

$$\Psi_1^{(2)} = -\frac{\mathrm{i}k}{2\pi} \int_0^{\infty} \mathrm{d}t \int_0^{\infty} \mathrm{d}s\, R_1^{(2)}(\tau - t, Y - s)\, \mathrm{e}^{-\mathrm{i}kYt} \int_L \mathrm{d}u\, \mathrm{e}^{\mathrm{i}kstu} u^{\frac{1}{2}} (1 - u)^{-\frac{3}{2}}$$

or, in explicit form,

$$\Psi_1^{(2)} = \frac{k^{\frac{5}{2}}}{16\pi^{\frac{3}{2}}} \mathrm{e}^{-\mathrm{i}\pi/4} \int_0^{\infty} \mathrm{d}t \int_0^{\infty} \mathrm{d}t_1 \int_C \mathrm{d}t_2 \int_C \mathrm{d}t_3\, B(\tau - t - t_1)\, B(\tau - t - t_2)\, \overline{B}(\tau - t - t_3)$$
$$\times (t_1 t_2 t_3)^{-\frac{3}{4}} (t_3 - t_1 - t_2)^{\frac{1}{2}} (t_3 - t - t_1 - t_2)^{-\frac{3}{2}} \Big\{ t_3 [t_1 (t_1 - t_2)^2 + t_2 (t_2 - t_3)^2$$
$$+ t_1 t_2 (2t_3 - t_1 - t_2)] - t_1^2 t_2 (t_1 + t_2)^{-\frac{3}{2}} (t_1 - t_2)^{\frac{1}{2}} (t_1 + t_2 + t_3)(2t_3 - t_1 - t_2)$$
$$\times F\left(-\frac{1}{2}, -\frac{1}{4}; \frac{3}{4}; \frac{t_2^2}{t_1^2}\right) \Big\}\, \mathrm{e}^{\mathrm{i}kY(t_3 - t - t_1 - t_2)}\, \theta(t_1 - t_2). \tag{3.6}$$

The asymptotic representation as $Y \to \pm\infty (\arg Y \leqslant 0)$

$$\Psi_1^{(2)} = D_1^{(2)\pm} Y^{\frac{1}{2}} + O(Y^{-\frac{1}{2}}) \quad (D_1^{(2)+} - D_1^{(2)-} \neq 0)$$

is such that it is impossible to obtain at this order non-zero $b_1^{(2)\pm}$ (see (2.8)) because there is no term of the required form ($\sim Y^{\frac{1}{2}} \ln(\frac{1}{2}\mu Y)$). Therefore, the strongest nonlinearity does not contribute to the evolution equation.

A simple analysis demonstrates that the absence of the term $\sim Y^{\frac{1}{2}} \ln(\frac{1}{2}\mu Y)$ in asymptotic representation is due to the fact that $\Phi = 0$ throughout the iteration considered. If we omit in (3.2) terms responsible for dissipation (or take the Prandtl number $\eta = 1$) and supercriticality, it is clear that $\Phi \equiv 0$ is a solution. This is the result of the same symmetry that has provided in Paper 1 the change of sign of the Landau constant at $\eta = 1$. This symmetry is an approximate one and there are a number of factors breaking it. First, these are the supercriticality and the dissipative effects. As they are represented by respective terms in (3.2) we call them the inner factors. Secondly, Φ may become non-zero if this is needed for matching to the outer solution. This is the outer factor. The linearity of the operator $\hat{\mathrm{N}}$ allows us to consider the contribution of each factor separately.

3.5. $O(\epsilon\mu^{\frac{3}{2}})$ *of the fundamental*

The addition to the fundamental due to the supercriticality is described by the equations

$$\hat{\mathrm{N}}_1 \Psi_1^{(3)} = \tfrac{1}{2}\mathrm{i}k\Phi_1^{(3)} - 2\mathrm{i}kJ^{(1)}W,$$
$$\hat{\mathrm{N}}_1 \Phi_1^{(3)} = -2\mathrm{i}kJ^{(1)}W.$$

Their solution is obtained using the procedure described in Appendix A and has a single asymptotic representation:

$$\Psi_1^{(3)} = -\tfrac{1}{2}J^{(1)}B(\tau)\, Y^{\frac{1}{2}} \ln^2(\tfrac{1}{2}\mu Y) - 2J^{(1)}B(\tau)\, Y^{\frac{1}{2}} \ln(\tfrac{1}{2}\mu Y) + O(Y^{-\frac{1}{2}} \ln^2(\tfrac{1}{2}\mu Y)),$$
$$\Phi_1^{(3)} = -2J^{(1)}B(\tau)\, Y^{\frac{1}{2}} \ln(\tfrac{1}{2}\mu Y) + O(Y^{-\frac{1}{2}} \ln(\tfrac{1}{2}\mu Y)).$$

Matching to (2.8) provides the *linear-in-B* contribution to $b_1^{(2)\pm}$

$$B(b_1^{(2)+} + b_1^{(2)-})_{\mathrm{L}} = \mathrm{i}\pi J^{(1)} B \tag{3.7}$$

(this is marked by subscript L) which determines the linear growth rate due to supercriticality. In principle, the harmonics generation will give rise to a nonlinear contribution to $b_1^{(2)\pm}$ proportional to $J^{(1)}$, but it will be small compared with (3.7) and cannot provide a competitive nonlinearity.

Thus, there are still two factors breaking the symmetry $\Phi = 0$ and, consequently, two ways of searching for the competitive nonlinearity: (i) the 'dissipative' one which provides a cubic nonlinearity and (ii) the 'non-dissipative' one which provides a quintic nonlinearity.

(i) *Cubic ('dissipative') nonlinearity*

Here we calculate the contribution to $b_1^{(2)\pm}$ due to the second inner factor, i.e. to dissipative terms on the right-hand sides of (3.2).

3.6(i). $O(\epsilon\kappa\mu^{-\frac{5}{2}})$ *of the fundamental*

$$\hat{\mathrm{N}}_1 \Psi_1^{(4)} = \tfrac{1}{2}\mathrm{i}k\Phi_1^{(4)} + \eta W''', \quad \hat{\mathrm{N}}_1 \Phi_1^{(4)} = (\eta - 1)\, W'''.$$

Using the obvious relation $0 = \mathscr{D}^3\hat{\mathrm{N}}_1 W = \hat{\mathrm{N}}_1 W''' + 3ikW'''$ we can easily integrate these equations, and after matching to (2.8) we obtain

$$\Psi_1^{(4)} = \frac{i(5\eta+1)}{18k} W''', \quad \Phi_1^{(4)} = \frac{i(\eta-1)}{3k} W''' \tag{3.8}$$

3.7(i). $O(\epsilon^2\kappa\mu^{-4})$ *of the second harmonic*

$$\hat{\mathrm{N}}_2 \Psi_2^{(2)} = ik\Phi_2^{(2)} + \frac{5\eta+1}{18}(2W'W^{\mathrm{iv}} - WW^{\mathrm{v}} - W''W''') + \eta\Psi_2^{(1)'''},$$

$$\hat{\mathrm{N}}_2 \Phi_2^{(2)} = \tfrac{1}{3}(\eta-1)(W'W^{\mathrm{iv}} - WW^{\mathrm{v}}) + (\eta-1)\,\Psi_2^{(1)'''}.$$

For further analysis we need $\Phi_2^{(2)}$ only. Its calculation is similar to that of $\Psi_2^{(1)}$ and provides

$$\Phi_2^{(2)} = \frac{i(\eta-1)}{192\pi} k^3 \int_0^\infty dt_1 \int_C dt_2 \int_C dt_3\, B(\tau-t_1-t_2)\, B(\tau-t_1-t_3)(t_2 t_3)^{-\frac{3}{2}}(t_2+t_3)^{\frac{1}{2}}$$
$$\times (2t_1+t_2+t_3)^{-\frac{3}{2}}(t_2-t_3)^2[(t_2-t_3)^2(t_2+t_3) + (2t_1+t_2+t_3)^3]$$
$$\times e^{-ikY(2t_1+t_2+t_3)}. \tag{3.9}$$

3.8(i). $O(\epsilon^2\kappa\mu^{-4})$ *of the zeroth harmonic*

$$\frac{\partial}{\partial\tau}\Psi_0^{(2)'} = -\frac{5\eta+1}{18}(\bar{W}'W''' - \bar{W}W^{\mathrm{iv}} - W\bar{W}^{\mathrm{iv}} + W'\bar{W}''')' + \eta\Psi_0^{(1)'''},$$

$$\frac{\partial}{\partial t}\Phi_0^{(2)'} = \frac{\eta-1}{3}(\bar{W}W^{\mathrm{iv}} + \mathrm{c.c.})' + (\eta-1)\,\Psi_0^{(1)'''}.$$

Integrating once with respect to Y we have

$$\left.\begin{aligned} \frac{\partial}{\partial\tau}\Psi_0^{(2)} &= \frac{5\eta+1}{18}(W^{\mathrm{iv}}\bar{W} - W'''\bar{W}' + \mathrm{c.c.}) + 2\eta\mathscr{D}^2|W'|^2, \\ \frac{\partial}{\partial\tau}\Phi_0^{(2)} &= \frac{\eta-1}{3}(W^{\mathrm{iv}}\bar{W} + W\bar{W}^{\mathrm{iv}} + 6\mathscr{D}^2|W'|^2). \end{aligned}\right\} \tag{3.10}$$

3.9(i). $O(\epsilon^3\kappa\mu^{-\frac{11}{2}})$ *of the fundamental*

At this order we must obtain the cubic-in-B contribution to $b_1^{(2)\pm}$. Let us write equations in concise form:

$$\hat{\mathrm{N}}_1 \Psi_1^{(5)} = \tfrac{1}{2}ik\Phi_1^{(5)} + R_1^{(5)}, \quad \hat{\mathrm{N}}_1 \Phi_1^{(5)} = R_2^{(5)}.$$

For deriving the evolution equation we need the asymptotic representations of $\Phi_1^{(5)}$ and $\Psi_1^{(5)}$ as $Y \to \pm\infty$, or more precisely the coefficient at $Y^{\frac{1}{2}}\ln(\frac{1}{2}\mu Y)$ in the expansion of $\Psi_1^{(5)}$ only. It is easy to see that $R_1^{(5)}$ and $R_2^{(5)}$ are localized and (A 13) and (A 14) are valid in this case. According to these, as $Y \to \pm\infty$

$$\Psi_1^{(5)} = C^{\pm}Y^{\frac{1}{2}} + D^{\pm}Y^{\frac{1}{2}}\ln(\tfrac{1}{2}\mu Y) + O(Y^{-\frac{1}{2}}\ln(\tfrac{1}{2}\mu Y)),$$

$$D^+ - D^- = \left(\frac{k}{\pi}\right)^{\frac{1}{2}} e^{i\pi/4}\int_{-\infty}^{\infty} dy \int_0^\infty dt\, t^{\frac{1}{2}} R_2^{(5)}(\tau-t, Y)\, e^{-ikYt}. \tag{3.11}$$

Matching to (2.8) provides the *cubic-in-B* contribution to $b_1^{(2)\pm}$ (see (3.1)):

$$B(b_1^{(2)+}+b_1^{(2)-})_{N3} = -(D^+-D^-) \tag{3.12}$$

(this is marked by subscript N3). Calculating the right-hand side requires $R_2^{(5)}$ only:

$$R_2^{(5)} = (\eta-1)\,\Psi_1^{(2)'''}+\mathrm{i}k[\Phi_0^{(2)''}W-\Phi_1^{(3)'}\Psi_0^{(1)'}+2\bar{\Phi}_1^{(3)''}\Psi_2^{(1)} +\bar{\Phi}_1^{(3)'}\Psi_2^{(1)'}-2\Phi_2^{(2)'}\bar{W}'-\Phi_2^{(2)''}\bar{W}].$$

One can see from (3.8)–(3.10) that $\Phi_l^{(m)}$ are proportional to $(\eta-1)$ so that $R_2^{(5)}\sim(\eta-1)$. Hence the right-hand side of (3.12) contains this factor too. It is convenient to separate contributions due to the zeroth and the second harmonics: $R_2^{(5)}=(\eta-1)(R_{20}+R_{22})$.

Using $\Psi_l^{(m)}$ and $\Phi_l^{(m)}$ derived earlier (in the form of integral representations) of which $R_2^{(5)}$ is built up, we obtain from (3.11) after very extended calculations for the zeroth and the second harmonics respectively

$$\begin{aligned}(D^+-D^-)_0 = {}& \frac{\mathrm{i}(\eta-1)}{48\pi}k^5\int_C \mathrm{d}t_1\int_C \mathrm{d}t_2\int_C \mathrm{d}t_3(t_1t_2t_3)^{-\frac{3}{2}}(t_1-t_2)^2(t_1-t_2-t_3)^{\frac{1}{2}}\\ &\times\{\tfrac{1}{2}\int_0^\infty \mathrm{d}t_4\,\bar{B}(\tau-2t_1+t_2+t_3-t_4)\,B(\tau-t_1+t_2)\,B(\tau-t_1+t_3-t_4)\\ &\times(t_1^4-6t_1^3t_2+12t_1^2t_2^2-6t_1t_2^3+t_2^4)-\bar{B}(\tau-2t_1+t_2+t_3)\\ &\times B(\tau-t_1+t_2)\,B(\tau-t_1+t_3)\,t_1t_2[(t_1-t_2)^3-2t_3(t_1-t_2)^2+2t_3^3]\}\,\theta(t_1-t_2-t_3),\end{aligned} \tag{3.13}$$

$$\begin{aligned}(D^+-D^-)_2 = {}& -\frac{\mathrm{i}(\eta-1)}{384\pi}k^5\int_0^\infty \mathrm{d}t_4\int_C \mathrm{d}t_1\int_C \mathrm{d}t_2\int_C \mathrm{d}t_3\,\bar{B}(\tau-2t_1+t_2+t_3+2t_4)\\ &\times B(\tau-t_1+t_2+t_4)\,B(\tau-t_1+t_3+t_4)(t_1t_2t_3)^{-\frac{3}{2}}(t_2+t_3)^{\frac{1}{2}}(t_2+t_3+2t_4)^{-\frac{3}{2}}\\ &\times(t_1-t_2-t_3-2t_4)^{\frac{1}{2}}(t_2-t_3)^2(2t_1-t_2-t_3-2t_4)\,[2t_1^4+(t_2+t_3+2t_4)^4\\ &+(t_2-t_3)^2(t_2+t_3)(t_2+t_3+2t_4)+2(t_1-t_2-t_3-2t_4)^3(t_1+t_2+t_3+2t_4)]\\ &\times\theta(t_1-t_2-t_3-2t_4).\end{aligned} \tag{3.14}$$

Considering the arguments of the amplitude functions B in (3.13) and (3.14) we can see that each product of B takes the form $B(\tau-x)\,B(\tau-y)\,\bar{B}(\tau-x-y)$. Using this fact we write a more compact form of (D^+-D^-):

$$D^+-D^- = \frac{\mathrm{i}(\eta-1)}{12}k^5\int_0^\infty \mathrm{d}s\,s^5\int_0^1 \mathrm{d}\sigma\,\sigma G(\sigma)\,B(\tau-s)\,B(\tau-\sigma s)\,\bar{B}(\tau-(1+\sigma)s). \tag{3.15}$$

The kernel $G(\sigma)$ is again represented as the sum of contributions due to the zeroth and the second harmonics: $G(\sigma) = G_0(\sigma)+G_2(\sigma)$ where

$$\left.\begin{aligned} G_0 = &-\left\{\tfrac{9}{8}\sigma\int_0^1 \mathrm{d}u\, u^{\frac{1}{2}}\left[16(\sigma+u)^{\frac{1}{2}}F_1\left(\tfrac{3}{2},-\tfrac{3}{2};-\tfrac{1}{2},1;\sigma,\frac{\sigma u}{\sigma+u}\right)-\sigma(1+\sigma u)^{\frac{1}{2}}\right.\right.\\ &\times F_1\left(\tfrac{3}{2},\tfrac{3}{2},-\tfrac{1}{2},4;\sigma,\frac{\sigma u}{1+\sigma u}\right)\Bigg]+2(1-\sigma)(1-4\sigma+6\sigma^2+\sigma^3)F\left(\tfrac{3}{2},\tfrac{1}{2};2;\sigma^2\right)\\ &+\frac{\sigma}{2}(1+\sigma)^{-\frac{3}{2}}\left[3(1+\sigma)^2(1+\sigma^2)F_1\left(\tfrac{3}{2},\tfrac{1}{2},-\tfrac{1}{2},3;\sigma,\frac{\sigma}{1+\sigma}\right)+3\sigma^2(1-\sigma^2)\right.\\ &\left.\left.\times F_1\left(\tfrac{3}{2},\tfrac{1}{2},\tfrac{1}{2},3;\sigma,\frac{\sigma}{1+\sigma}\right)+(1-3\sigma+9\sigma^2-6\sigma^3+\sigma^4)F_1\left(\tfrac{3}{2},\tfrac{1}{2},\tfrac{3}{2},3;\sigma,\frac{\sigma}{1+\sigma}\right)\right]\right\},\\ G_2 = &\frac{1+\sigma}{\pi}\int_0^1\frac{\mathrm{d}t\, t^{-\frac{1}{4}}(1-t)^{\frac{1}{2}}(1-\sigma t)^{\frac{1}{2}}}{(1+\sigma t)^{\frac{3}{2}}(1-\sigma^2 t)^{\frac{3}{2}}}\left\{\left[2\left(\frac{1-\sigma^2 t}{1-\sigma t}\right)^4+(1-\sigma)^4\left(\frac{1+\sigma t}{1-\sigma t}\right)^4\right.\right.\\ &\left.+2\sigma^3\left(\frac{1-t}{1-\sigma t}\right)^3\frac{2-\sigma+\sigma t-2\sigma^2 t}{1-\sigma t}\right]F\left(-\tfrac{1}{2},-\tfrac{1}{4};\tfrac{3}{4};\sigma^2t^2\right)\\ &\left.+(1-\sigma)^4\frac{1+\sigma t}{1-\sigma t}\left[(1-\sigma^2t^2)^{\frac{1}{2}}-2\sigma t F\left(\tfrac{1}{2},\tfrac{1}{4};\tfrac{5}{4};\sigma^2t^2\right)\right]\right\}. \end{aligned}\right\} \quad (3.16)$$

Here F and F_1 are the hypergeometric functions of a single or two variables, respectively (Erdélyi *et al.* 1953)

$$F(a,b;c;z) = \sum_{n=0}^{\infty}\frac{\Gamma(a+n)\,\Gamma(b+n)\,\Gamma(c)}{\Gamma(a)\,\Gamma(b)\,\Gamma(c+n)}\frac{z^n}{n!},$$

$$F_1(a,b,c,d;x,y) = \sum_{m,n=0}^{\infty}\frac{\Gamma(a+m+n)\,\Gamma(b+m)\,\Gamma(c+n)\,\Gamma(d)}{\Gamma(a)\,\Gamma(b)\,\Gamma(c)\,\Gamma(d+m+n)}\frac{x^m y^n}{m!\,n!}.$$

So we have obtained the cubic-in-B ('dissipative') contribution to $b_1^{(2)\pm}$.

(ii) *Quintic ('non-dissipative') nonlinearity*

Here we calculate a quintic contribution to $b_1^{(2)\pm}$ which, as we shall see, is due to the outer factor: it is the matching to an outer solution that breaks the symmetry $\Phi = 0$.

3.6(ii). $O(\epsilon^3\mu^{-\frac{5}{2}})$ *of the third harmonic*

$$\hat{\mathrm{N}}_3\,\Psi_3^{(1)} = \tfrac{3}{2}\mathrm{i}k\Phi_3^{(1)}+\mathrm{i}k(\Psi_2^{(1)\prime\prime}W-3\Psi_2^{(1)\prime}W'+2\Psi_2^{(1)}W''),\quad \hat{\mathrm{N}}_3\,\Phi_3^{(1)} = 0.$$

The right-hand sides have a single asymptotic representation and, therefore, matching to (3.11) gives $\Phi_3^{(1)} = 0$, while for $\Psi_3^{(1)}$ we obtain

$$\begin{aligned} \Psi_3^{(1)} = &-\frac{\kappa^{\frac{5}{2}}}{16\pi^{\frac{3}{2}}}\,\mathrm{e}^{\mathrm{i}\pi/4}\int_0^\infty \mathrm{d}t\int_0^\infty \mathrm{d}t_1\int_C \mathrm{d}t_2\int_C \mathrm{d}t_3\, B(\tau-t-t_1)\,B(\tau-t-t_2)\,B(\tau-t-t_3)\\ &\times t_1^{\frac{1}{2}}t_2^{-\frac{1}{2}}t_3^{-\frac{3}{2}}(t_1+t_2)^{-\frac{3}{2}}(t_1-t_2)^{\frac{1}{2}}(t_1+t_2+t_3)^{\frac{1}{2}}(3t+t_1+t_2+t_3)^{-\frac{3}{2}}(t_1+t_2-t_3)\\ &\times(t_1+t_2-2t_3)\,F\left(-\frac{1}{2},-\frac{1}{4};\frac{3}{4};\frac{t_2^2}{t_1^2}\right)\mathrm{e}^{-\mathrm{i}kY(3t+t_1+t_2+t_3)}\,\theta(t_1-t_2). \end{aligned} \quad (3.17)$$

3.7(ii). $O(\epsilon^4\mu^{-4})$ *of the zeroth harmonic*

In fourth-order iterations in amplitude we have to calculate only the corresponding contributions to Φ. In the case of a zeroth harmonic, equations obtainable from (3.1) are not interrelated so that Ψ and Φ can both be calculated and matched separately. The equation for Φ,

$$\frac{\partial}{\partial \tau}\Phi_0^{(3)\prime} = 0,$$

has a solution $\Phi_0^{(3)\prime} = f(Y)$. The difference of $f(Y)$ from zero is equivalent to choosing another density profile of an unperturbed flow; this will not affect the final result (the nonlinear term), however. Hence, without any loss of generality, we let $f(Y) = 0$ and $\Phi_0^{(3)} = 0$.

3.8(ii). $O(\epsilon^4\mu^{-4})$ *of the second harmonic*

The equations that follow from (3.2) can be written conveniently as

$$\left.\begin{aligned} \hat{\mathrm{N}}_2\Psi_2^{(3)} - \mathrm{i}k\Phi_2^{(3)} &= \mathrm{i}kR_N \equiv \mathrm{i}k(R_{11}+R_{02}+R_{31}),\\ \hat{\mathrm{N}}_2\Phi_2^{(3)} &= 0, \end{aligned}\right\} \tag{3.18}$$

where $R_{11} = \Psi_1^{(2)\prime\prime}W - 2\Psi_1^{(2)\prime}W' + \Psi_1^{(2)}W''$, $R_{02} = 2(\Psi_0^{(1)\prime\prime}\Psi_2^{(1)} - \Psi_0^{(1)\prime}\Psi_2^{(1)\prime})$,

$$R_{31} = 3\Psi_3^{(1)}\overline{W}'' - 2\Psi_3^{(1)\prime}\overline{W}' - \Psi_3^{(1)\prime\prime}\overline{W}.$$

Let us write the solution as the sum $\Psi_2^{(3)} = \Psi_{2\mathrm{N}}^{(3)} + \Psi_{2\mathrm{L}}^{(3)}$ of a particular solution ($\hat{N}_2\Psi_{2\mathrm{N}}^{(3)} = \mathrm{i}kR_{\mathrm{N}}$) and a general solution of the homogeneous system $\Psi_{2\mathrm{L}}^{(3)}$. We obtain (cf. (3.3))

$$\left.\begin{aligned} \Phi_2^{(3)} &= \frac{\mathrm{e}^{3\mathrm{i}\pi/4}}{2(2\pi k)^{\frac{1}{2}}}\int_C \mathrm{d}t\, t^{-\frac{3}{2}}C(\tau-t)\,\mathrm{e}^{-2\mathrm{i}kYt},\\ \Psi_{2\mathrm{L}}^{(3)} &= \frac{\mathrm{e}^{3\mathrm{i}\pi/4}}{4(2\pi k)^{\frac{1}{2}}}\int_C \mathrm{d}t\, t^{-\frac{3}{2}}[C(\tau-t)\,g_2(t) + D(\tau-t)]\,\mathrm{e}^{-2\mathrm{i}kYt},\\ g_m &= \psi(\tfrac{3}{2}) - \ln\frac{2mkt}{\mu} - \tfrac{3}{2}\mathrm{i}\pi, \quad m = 1,2,3,\ldots, \end{aligned}\right\} \tag{3.19}$$

where $\psi(z)$ is a digamma function (Abramowitz & Stegun 1964) and $C(\tau)$ and $D(\tau)$ are arbitrary functions which tend to zero as $\tau \to -\infty$ (for details of the calculations see Appendix A). As $Y \to \pm\infty$,

$$\Psi_{2\mathrm{N}}^{(3)} = M^{\pm}Y^{\frac{1}{2}} + O(Y^{-\frac{1}{2}}),$$

where
$$M^+ - M^- = \frac{(2k)^{\frac{3}{2}}}{\pi^{\frac{1}{2}}}\mathrm{e}^{3\mathrm{i}\pi/4}\int_{-\infty}^{\infty}\mathrm{d}Y\int_0^{\infty}\mathrm{d}t\, R_{\mathrm{N}}(\tau-t, Y)t^{\frac{1}{2}}\,\mathrm{e}^{-2\mathrm{i}kYt} \neq 0. \tag{3.20}$$

It is easy to see that the particular solution does not match to (2.10) and, therefore, $\Psi_{2\mathrm{L}}^{(3)}$ should be added to it. Hence, the asymptotic representation of $\Psi_2^{(3)}$ has the form

$$\Psi_2^{(3)} = [C(\tau)\ln(\tfrac{1}{2}\mu Y) + D(\tau) + M^{\pm}(\tau)]\,Y^{\frac{1}{2}} + O(Y^{-\frac{1}{2}}\ln(\tfrac{1}{2}\mu Y))$$

For matching, in (2.10) it is necessary to put $c^{\pm} = \epsilon^4\mu^{-\frac{9}{2}}c_2^{\pm}$. We obtain

$$c_2^+ = \mathrm{i}c_2^- = C(\tau), \quad c_2^+q^+ = D(\tau) + M^+(\tau), \quad \mathrm{i}c_2^-q^- = D(\tau) + M^-(\tau) - \mathrm{i}\pi C(\tau),$$

whence
$$C(\tau) = \frac{M^+ - M^-}{q^+ - q^- - \mathrm{i}\pi}. \tag{3.21}$$

As has been pointed out previously, in view of the symmetry of our model we have $q^+ = q^-$; therefore

$$C(\tau) = \frac{\mathrm{i}}{\pi}(M^+ - M^-) \tag{3.22}$$

which, together with (3.19) and (3.20), fully defines $\Phi_2^{(3)} \neq 0$.

Thus, it is the matching to the outer solution that has led to breaking the symmetry $\Phi = 0$. As a result, in the next iteration (of the fundamental), the right-hand side of the equation for Φ will already not have a single asymptotic representation, and this will lead to a nonlinear contribution to the evolution equation.

In accordance with (3.18), it is convenient to write

$$M^+ - M^- = (M^+ - M^-)_{11} + (M^+ - M^-)_{02} + (M^+ - M^-)_{31}. \tag{3.23}$$

Simple but cumbersome calculations yield

$$\left.\begin{aligned}
(M^+ - M^-)_{11} = {} & \left(\frac{k^9}{8\pi^3}\right)^{\frac{1}{2}} \mathrm{e}^{\mathrm{i}\pi/4} \int_0^\infty \mathrm{d}t \int_0^\infty \mathrm{d}t_4 \int_C \mathrm{d}t_1 \int_C \mathrm{d}t_2\, B(\tau - t - t_1) \\
& \times B(\tau - t - t_2)\, B(\tau + t + t_1 + t_2 - t_4) \\
& \times \bar{B}(\tau - t - t_4)\, (t_1 t_2 t_4)^{-\frac{3}{2}} (t_4 - t_1 - t_2)^{\frac{1}{2}} (t_4 - t - t_1 - t_2)^{\frac{1}{2}} \\
& \times \Big\{ t_4[t_1(t_4 - t_1)^2 + t_2(t_4 - t_2)^2 + t_1 t_2(2t_4 - t_1 - t_2)] - t_1^2 t_2 (t_1 + t_2)^{-\frac{3}{2}} \\
& \times (t_1 - t_2)^{\frac{1}{2}} (t_1 + t_2 + t_4)(2t_4 - t_1 - t_2)\, F\left(-\frac{1}{2}, -\frac{1}{4}; \frac{3}{4}; \frac{t_2^2}{t_1^2}\right)\Big\} \\
& \times \theta(t_1 - t_2)\, \theta(t_4 - t_1 - t_2 - t) \int_C \mathrm{d}x\, x^{\frac{1}{2}} (1 - x^2)^{-\frac{3}{2}} \theta(z - x), \\
z = {} & \frac{t}{t_4 - t - t_1 - t_2};
\end{aligned}\right\} \tag{3.24}$$

$$\begin{aligned}
(M^+ - M^-)_{02} = {} & \left(\frac{2k^9}{\pi^3}\right)^{\frac{1}{2}} \mathrm{e}^{\mathrm{i}\pi/4} \int_0^\infty \mathrm{d}t \int_0^\infty \mathrm{d}t_4 \int_C \mathrm{d}t_1 \int_C \mathrm{d}t_2\, B(\tau - t - t_1)\, B(\tau - t - t_2) \\
& \times B(\tau + t + t_1 + t_2 - t_4) \bar{B}(\tau - t - t_4)\, (t t_1)^{\frac{1}{2}} (t_2 t_4)^{-\frac{3}{2}} (t_4 - 2t - t_1 - t_2)^{-\frac{1}{2}} (t_1 + t_2)^{-\frac{3}{2}} \\
& \times (t_1 - t_2)^{\frac{1}{2}} (2t + t_1 + t_2)(t_1 + t_2 + t)\, F\left(-\frac{1}{2}, -\frac{1}{4}; \frac{3}{4}; \frac{t_2^2}{t_1^2}\right) \\
& \times \theta(t_4 - 2t - t_1 - t_2)\, \theta(t_1 - t_2);
\end{aligned} \tag{3.25}$$

$$\begin{aligned}
(M^+ - M^-)_{31} = {} & \left(\frac{k^9}{2\pi^3}\right)^{\frac{1}{2}} \mathrm{e}^{\mathrm{i}\pi/4} \int_0^\infty \mathrm{d}t \int_0^\infty \mathrm{d}t_4 \int_0^\infty \mathrm{d}t_1 \int_C \mathrm{d}t_2 \int_C \mathrm{d}t_3\, B(\tau - t - t_1 - t_3) \\
& \times B(\tau - t - t_2 - t_3) B(\tau + t + t_1 + t_2 + 2t_3 - t_4)\, \bar{B}(\tau - t - t_4)(t t_1)^{\frac{1}{2}} t_2^{-\frac{1}{2}} t_4^{-\frac{3}{2}} \\
& \times (t_4 - 2t - t_1 - t_2 - 3t_3)^{-\frac{3}{2}} (t_1 + t_2)^{-\frac{3}{2}} (t_1 - t_2)^{\frac{1}{2}} (t_4 - 2t - 3t_3)^{\frac{1}{2}} (t_4 - 2t)^{-\frac{3}{2}} \\
& \times (t_4 - 2t - 2t_1 - 2t_2 - 3t_3)(2t_4 - 4t - 3t_1 - 3t_2 - 6t_3) \\
& \times (t_4^2 - t^2)\, F\left(-\frac{1}{2}, -\frac{1}{4}; \frac{3}{4}; \frac{t_2^2}{t_1^2}\right) \theta(t_4 - 2t - t_1 - t_2 - 3t_3)\, \theta(t_1 - t_2).
\end{aligned} \tag{3.26}$$

3.9 (ii). $O(\epsilon^5\mu^{-\frac{11}{2}})$ *of the fundamental*

At this order we must obtain the quintic-in-B contribution to $b_1^{(2)\pm}$. The equations resulting from (3.2) can be written in concise form

$$\hat{\mathrm{N}}_1\,\Psi_1^{(6)}-\tfrac{1}{2}\mathrm{i}k\Phi_1^{(6)} = R_1,$$
$$\hat{\mathrm{N}}_1\,\Phi_1^{(6)} = -\mathrm{i}k(\Phi_2^{(3)\prime\prime}\overline{W}+2\Phi_2^{(3)\prime}\overline{W}') \equiv -\mathrm{i}kR.$$

The asymptotic representation of $\Psi_1^{(6)}$ is calculated rather simply:

$$\Psi_1^{(6)} = m_1^{(6)\pm}Y^{\frac{1}{2}}+N^{\pm}Y^{\frac{1}{2}}\ln(\tfrac{1}{2}\mu Y)+O(Y^{-\frac{1}{2}}\ln(\tfrac{1}{2}\mu Y)),$$

where $$N^{+}-N^{-} = \frac{k^{\frac{3}{2}}}{\pi^{\frac{1}{2}}}\,\mathrm{e}^{-\mathrm{i}\pi/4}\int_{-\infty}^{\infty}\mathrm{d}Y\int_0^{\infty}\mathrm{d}t R(\tau-t,Y)\,t^{\frac{1}{2}}\,\mathrm{e}^{-\mathrm{i}kYt}.$$

Using (3.3) and (3.19) we find

$$R = -\frac{k}{2^{\frac{3}{2}}\pi}\int_C\mathrm{d}t_1\int_C\mathrm{d}t_2C(\tau-t_1)\,\overline{B}(\tau-t_2)\,t_1^{-\frac{1}{2}}t_2^{-\frac{3}{2}}(t_1-t_2)\,\mathrm{e}^{-\mathrm{i}kY(2t_1-t_2)}$$

and obtain

$$N^{+}-N^{-} = \frac{8\pi k^{\frac{3}{2}}}{\Gamma^2(\frac{1}{4})}\,\mathrm{e}^{-\mathrm{i}\pi/4}\int_0^{\infty}\mathrm{d}t\,t^{\frac{1}{2}}C(\tau-t)\,\overline{B}(\tau-2t), \tag{3.27}$$

where $\Gamma(z)$ is the gamma-function. Therefore, according to (3.1), the *quintic contribution* to $b_1^{(2)\pm}$ is

$$B(b_1^{(2)+}+b_1^{(2)-})_{N5} = -(N^{+}-N^{-}), \tag{3.28}$$

where, according to (3.27) and (3.22), we have

$$N^{+}-N^{-} = \frac{8k^{\frac{3}{2}}}{\Gamma^2(\frac{1}{4})}\,\mathrm{e}^{\mathrm{i}\pi/4}\int_0^{\infty}\mathrm{d}t\,t^{\frac{1}{2}}\overline{B}(\tau-2t)\,[M^{+}(\tau-t)-M^{-}(\tau-t)]. \tag{3.29}$$

The jump $M^{+}-M^{-}$ is defined by formulae (3.23)–(3.26).

Analysis of the arguments of the amplitude functions in (3.24)–(3.26) shows that in each expression, only three of four arguments are independent. This allows us to write $M^{+}-M^{-}$ in universal form:

$$M^{+}-M^{-} = \left(\frac{8k^9}{\pi^3}\right)^{\frac{1}{2}}\mathrm{e}^{\mathrm{i}\pi/4}\int_0^{\infty}\mathrm{d}t\,t^{\frac{7}{2}}\int_0^1\mathrm{d}x\,x\int_0^1\mathrm{d}y\,G(x,y)\,B(\tau-t)$$
$$\times B(\tau-xt)\,B(\tau-xyt)\,\overline{B}(\tau-(1+x+xy)\,t). \tag{3.30}$$

(In view of its awkwardness, the explicit form of the kernel $G(x,y)$ is given in Appendix B.) Substitution of (3.30) into (3.29) and a little manipulation reduce (3.29) to the form

$$\left.\begin{aligned} N^{+}-N^{-} &= \frac{\mathrm{i}2^{\frac{9}{2}}k^6}{\pi^{\frac{3}{2}}\Gamma^2(\frac{1}{4})}\int_0^{\infty}\mathrm{d}t\,t^5\int_0^1\mathrm{d}x\int_0^1\mathrm{d}y\int_0^1\mathrm{d}z H(x,y,z)\,B(\tau-t)\,B(\tau-xt)\\ &\quad\times B(\tau-xyt)\,\overline{B}(\tau-2xyzt)\,\overline{B}(\tau-(1+x+xy-2xyz)t),\\ H(x,y,z) &= (x^5y^3z)^{\frac{1}{2}}(1-xyz)^{\frac{3}{2}}G\left(x\frac{1-yz}{1-xyz},y\frac{1-z}{1-yz}\right).\end{aligned}\right\} \tag{3.31}$$

The kernels $H(x,y,z)$ and $G(x,y)$ are *positive* (see Appendix B).

So we have calculated three main contributions to $b_1^{(2)\pm}$: a linear one (3.7), a cubic one (3.12), and a quintic one (3.28) and (3.31). Generally they are of different orders of magnitude ($O(1)$, $O(\epsilon^2\kappa\mu^{-7})$, and $O(\epsilon^4\mu^{-7})$ respectively) and which of them will play the leading role in the evolution equation is determined by relative values of the small parameters ϵ, κ, and μ.

When the amplitude of perturbation is very small ($\epsilon \ll \min[\mu^{\frac{7}{4}}, (\mu^7/\kappa)^{\frac{1}{2}}]$), the linear contribution (3.7) will be a leading one and the MSC (2.12) will provide the evolution equation of the linear theory,

$$\frac{\mathrm{d}B}{\mathrm{d}\tau} = \Gamma B, \quad \Gamma = -kJ^{(1)},$$

describing an exponential growth of amplitude. Sooner or later this growth will make one of the two nonlinearities become a competitive one.

The cubic nonlinearity becomes competitive as $A \sim (\mu^7/\kappa)^{\frac{1}{2}}$, and the quintic one does so as $A \sim \mu^{\frac{7}{4}}$. Therefore if $\mu > \kappa^{\frac{2}{7}}$ the quintic nonlinearity becomes competitive earlier (at smaller A) and the cubic nonlinearity will be small compared with it as the amplitude increases. The evolution in this case will be described by an equation that is obtained by simultaneously including the contributions (3.7) and (3.28) in the MSC (2.12):

$$\frac{\mathrm{d}B}{\mathrm{d}\tau} = -kJ^{(1)}B + N, \tag{3.32}$$

$$N = \frac{2^{\frac{9}{2}}k^7}{\pi^{\frac{5}{2}}\Gamma^2(\frac{1}{4})}\int_0^\infty \mathrm{d}t\, t^5 \int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y \int_0^1 \mathrm{d}z\, H(x,y,z)\, B(\tau-t)\, B(\tau-xt)$$
$$\times B(\tau-xyt)\, \bar{B}(\tau-2xyzt)\, \bar{B}(\tau-(1+x+xy-2xyz)\,t). \tag{3.33}$$

To derive it we must formally take $\epsilon = \mu^{\frac{7}{4}}$.

At the smaller supercriticality ($\mu < \kappa^{\frac{2}{7}}$) the cubic non-linearity becomes competitive earlier than the quintic one and the evolution is described by

$$\frac{\mathrm{d}B}{\mathrm{d}\tau} = -kJ^{(1)}B + \frac{\eta-1}{12\pi}k^6\int_0^\infty \mathrm{d}s\, s^5 \int_0^1 \mathrm{d}\sigma\, \sigma G(\sigma)\, B(\tau-s)\, B(\tau-\sigma s)\, \bar{B}(\tau-s-\sigma s), \tag{3.34}$$

and for its derivation we must take $\epsilon = (\mu^7/\kappa)^{\frac{1}{2}}$. But if in the process of perturbation development its amplitude achieves the $A \sim \kappa^{\frac{1}{2}}$ level, the quintic nonlinearity becomes the leading one and for description of the further evolution equation (3.32) will be appropriate.

4. The evolution equation: analysis and solutions

Thus, we have derived the amplitude equation which – depending on the region of parameters – may have one of two forms. Let us return in (3.34) and (3.32) to the 'physical' variables $t(=\tau/\mu)$ and $A(=\epsilon B)$:

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \gamma_{\mathrm{L}}A + \frac{\eta-1}{12\pi}\kappa k^6\int_0^\infty \mathrm{d}s\, s^5 \int_0^1 \mathrm{d}\sigma\, \sigma\, G(\sigma)\, A(t-s)\, A(t-\sigma s)\, \bar{A}(t-s-\sigma s) \tag{4.1}$$

in Region $II^{(3)}$; and

$$\frac{dA}{dt} = \gamma_L A + \lambda \int_0^\infty ds\, s^5 \int_0^1 dx \int_0^1 dy \int_0^1 dz\, H(x,y,z)\, A(t-s)\, A(t-sx) \times A(t-sxy)\, \bar{A}(t-2sxyz)\, \bar{A}(t-(1+x+xy-2xyz)\, s) \quad (4.2)$$

in Region $II^{(5)}$. Here $\gamma_L = k(\frac{1}{4}-J)$ is a growth rate of linear theory, and λ = constant.

Now we discuss the properties of the equations and solutions obtained.

4.1. *Cubic nonlinearity*

(i) As the nonlinear term in (4.1) is fully due to the harmonic interaction inside the CL, it does not depend on the arrangement of the flow as a whole, i.e. on the choice of the flow model. The flow model determines only the value of the wavenumber k and the coefficient in the relation between the growth rate γ_L and the supercriticality $(\frac{1}{4}-J)$. A similar situation occurs in the case of a viscous CL (see Paper 1).

(ii) Equation (4.1) is invariant to the translation in time $t \rightarrow t+t_0$ (this property is evidently due to the character of the problem, i.e. to 'switching on' the perturbation at $t \rightarrow -\infty$). Therefore (4.1) will have a variety of solutions which can be obtained one from another by the translation in time. At the initial stage of the perturbation development where the nonlinear term in (4.1) may be neglected, the solution is $A = A_0\, e^{\gamma_L t}$. By a suitable choice of A_0 we fix the time zero and thereby the unique solution.

Equation (4.1) may be reduced to a 'universal' form that does not depend on A_0. Let us put

$$A(t) = A_0\, b(t)\, e^{\gamma_L t}$$

Then, $b(t) \rightarrow 1$ as $t \rightarrow -\infty$. Now, we introduce the 'logarithmic' time

$$T = \kappa \frac{|\eta-1|}{12\pi} \left(\frac{k}{2\gamma_L}\right)^6 A_0^2\, e^{2\gamma_L t} \quad (0 \leqslant T < \infty)$$

and considering b to be a function of T we obtain the equation

$$\frac{db}{dT} = \tfrac{1}{2}\zeta \int_0^1 d\sigma\, K(\sigma) \int_0^1 dx\, x^5\, e^{-x} b(T\, e^{-x})\, b(T\, e^{-x/(\sigma+1)})\, b(T\, e^{-\sigma x/(\sigma+1)}) \quad (4.3)$$

together with the initial condition $b(0) = 1$. Here

$$\zeta = \operatorname{sgn}(\eta-1), \quad K(\sigma) = \sigma G(\sigma)\,(1+\sigma)^{-6}.$$

(iii) Equation (4.1) is an integro-differential one in contrast to those usually encountered in the theory of nonlinear stability of shear flows. Apparently, such an equation arises inevitably when the nonlinearity is a competitive one in the unsteady CL regime (see also Hickernell 1984). The rate of the amplitude variation is determined in this regime by the whole history of the perturbation development rather than by the instantaneous amplitude magnitude. This makes it too difficult not only to search for the solution but also to determine its asymptotic properties. Generally, the analytical methods of investigation such an equation are very much restricted and we need to use numerical calculations.

Now, let us consider qualitatively (as far as it is possible) the behaviour of the solutions of the evolution equation (4.1).

The first stage: non-competitive nonlinearity

At the initial stage (for large negative t) the linear term dominates in the right-hand side of (4.1). The amplitude grows exponentially and the nonlinearity provides only small corrections to this law. Substituting $A = A_0\, e^{\gamma_L t}$ into (4.1) we obtain

$$\frac{dA}{dt} = \gamma_L A + aA^3,$$

where the Landau constant a is

$$a = \frac{10\kappa}{\pi}(\eta - 1)\left(\frac{k}{2\gamma_L}\right)^6 \int_0^1 K(\sigma)\, d\sigma.$$

In the case of non-competitive nonlinearity a may be calculated using essentially rather simple calculations compared with §3. We have done so with the purpose of controlling the expression for the kernel $G(\sigma)$. These calculations show that two equalities must be satisfied (for the zeroth and the second harmonics respectively):

$$\int_0^1 K_0(\sigma)\, d\sigma = \frac{1}{20}\left(9 - \frac{217\pi}{64}\right), \quad \int_0^1 K_2(\sigma)\, d\sigma = \frac{1}{20}\left(\frac{55\pi}{16} - \frac{80}{9}\right).$$

Here $K_m(\sigma) = \sigma G_m(\sigma)(1+\sigma)^{-6}$, $m = 0, 2$; $G_m(\sigma)$ are defined by (3.16). Unfortunately, we were unable to calculate these integrals (and thereby run a check as described above) analytically. That is why this integration has been made numerically and has verified the correctness of formulae (3.16) for the kernel of the evolution equation.

The numerical calculation demonstrates that $K_0 < 0$ and $K_2 > 0$ for $0 \leqslant \sigma \leqslant 1$ and their sum (i.e. $K(\sigma)$) changes its sign at $\sigma = 0.45$ (figure 3). The integral $\int_0^1 K\, d\sigma > 0$ and therefore the Landau constant is a positive one when $\eta - 1 > 0$ and is a negative one when $\eta - 1 < 0$. Note that the nonlinear evolution equation for the viscous CL regime obtained in Paper 1 has the same property. One can consider the nonlinear equation (4.1) as a 'continuation' of the 'viscous' evolution equation mentioned above into the domain $\gamma_L > \nu^{\frac{1}{3}}$.

However, there is an essential distinction between these two equations. In a 'viscous' equation the nonlinear term is an algebraic function of $A(t)$ and therefore its magnitude is determined only by the instantaneous value of the amplitude and its sign is determined by the sign of $(\eta - 1)$. That is why the character of the perturbation evolution can easily be predicted in this case: namely, if the sign of $(\eta - 1)$ is stabilizing, the amplitude tends to some constant value; otherwise, the amplitude growth is accelerated more and more and finally becomes a 'burst-like' one. But in the unsteady CL regime both the amplitude and the sign of the nonlinear term are determined (except for the sign of $\eta - 1$) by the whole history of perturbation development, i.e. by the character of $A(t)$ behaviour and, in addition, there is competition between the 'near past' and the 'remote past' owing to the alternating-sign nature of $K(\sigma)$. This results in the nearly total impossibility of predicting qualitatively the perturbation evolution. As we shall demonstrate below, independently of the sign of $(\eta - 1)$ the perturbation growth is unbounded, though its character depends on this sign.

The third stage: the dominant nonlinearity

Suppose that the perturbation is so large that one can retain the nonlinear term only on the right-hand side of (4.1). A simple analysis demonstrates the possibility of a power-like solution

$$A = A_*(t_0 - t)^{-\frac{7}{2}}. \tag{4.4}$$

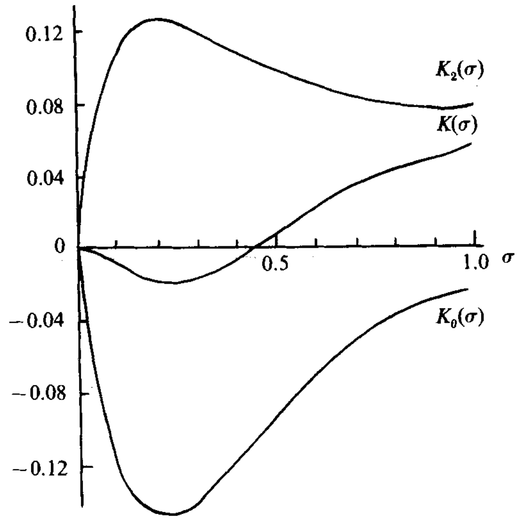

FIGURE 3. The kernel of the nonlinear evolutionary equation (4.3), $K(\sigma) = K_0(\sigma)+K_2(\sigma)$.

Substitution into (4.1) yields

$$A_*^2 = \frac{42\pi}{(\eta-1)\kappa}k^{-6}I_3^{-1}; \quad I_3 = \int_0^1 \mathrm{d}\sigma\, K(\sigma) \int_0^1 \frac{\mathrm{d}x\, x^5(1-x)^{\frac{7}{2}}}{\left(1-\dfrac{x}{1+\sigma}\right)^{\frac{7}{2}}\left(1-\dfrac{\sigma x}{1+\sigma}\right)^{\frac{7}{2}}} \approx 1.7\times 10^{-4}. \quad (4.5)$$

As I_3 proves to be positive (in spite of $K(\sigma) < 0$ when $\sigma < 0.45$) it follows from (4.5) that the 'burst-like' solution (4.4) is possible (i.e. $A_*^2 > 0$) only when $\eta > 1$. It is the case in which the nonlinear term at the stage of a non-competitive nonlinearity demonstrates a tendency to destabilization.

Thus the solution (4.4) can be considered to be a 'continuation' of the 'burst-like' one ($A \sim (t_0-t)^{-\frac{1}{2}}$) in the viscous CL regime obtained in Paper 1 in the domain $\nu^{\frac{2}{3}} < \epsilon (< \nu^{\frac{1}{2}})$. At the transition from the viscous CL regime ($\epsilon < \nu^{\frac{2}{3}}$) to the unsteady one ($\epsilon > \nu^{\frac{2}{3}}$) the 'burst-like' character of the perturbation growth is maintained and this growth is even accelerated.

When $\eta < 1$. we have not obtained any analytic results. Numerical solution results are described in the following sub-section.

The second stage: equipartition

Now we investigate the most difficult case, where both terms on the right-hand side of (4.1) are equally important. For the numerical solution we take equation (4.3) which *has a universal character and does not contain any parameter* (except for the sign of $(\eta-1)$).

The numerical solution of (4.3) for $\eta > 1$ confirms the qualitative conclusion about a monotonic growth of amplitude and transition into the 'burst-like' regime (figure 4). Indeed, the amplitude growth is unlimited as $T \rightarrow T_0 (T_0 \approx 41)$ according to the law $b(T) = 1.4\times 10^7 (41-T)^{-\frac{7}{2}}$.

When $\eta < 1$, the character of the evolution is more complicated. It is easy to see that while as $b(T)$ varies slowly enough (as it does at the initial stage), the main contribution to the integral in (4.3) is due to $x \approx 5$ (i.e. to $b(0)$ in fact). Consequently, the right-hand side of (4.3) is nearly constant (negative) for a relatively long time. In

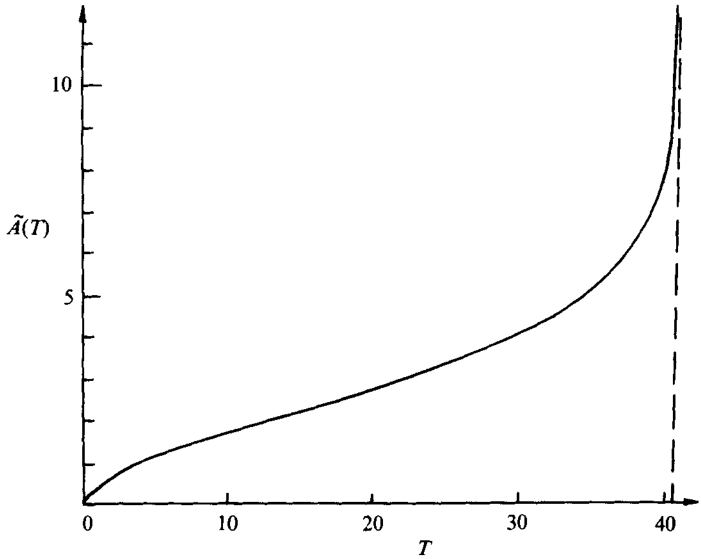

FIGURE 4. The amplitude development with 'logarithmic' time for the Prandtl number $\eta > 1$. $\tilde{A}(T) \equiv \log_{10}(1+T^{\frac{1}{2}}|b(T)|)\,\mathrm{sgn}\,b(T)$. As $T \to 0$, $\tilde{A} \sim T^{\frac{1}{2}}$

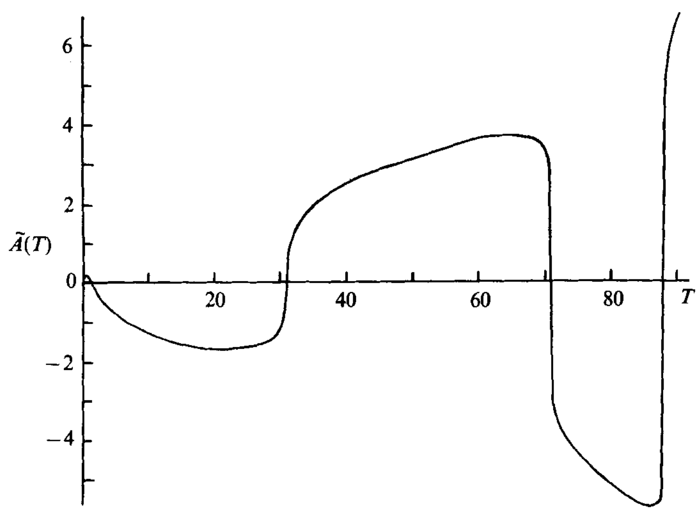

FIGURE 5. The same as figure 4 but for the Prandtl number $\eta < 1$.

view of this, $b(T)$ changes its sign suddenly (at $T \approx 1.3$) and then grows in absolute value to be nearly linear in time. As time elapses, the character of the $b(T)$ variation changes: $b(T)$ begins to oscillate with a fast rising amplitude (figure 5).† After a few oscillations the amplitude rises by some orders of magnitude and becomes significantly greater than $\nu^{\frac{1}{2}}$, and we already need to take into account the nonlinearity A^5 (see the next subsection).

† This does not mean that, when A passes through a turning point, the balance in the critical layer changes to a nonlinear one. One must simply redefine the measure of unsteadiness as $\gamma \sim |A^{-1}\,\mathrm{d}^2A/\mathrm{d}t^2|^{\frac{1}{2}}$ at such a point (for example).

It is interesting to see how the streamline pattern changes when $\eta < 1$. Inside the critical layer there exists a region of closed streamlines under the separatrix with width $\Delta Y \sim \epsilon^{\frac{1}{2}}\mu^{-\frac{3}{4}}$ (so-called 'cat's eyes') so that as $Y \sim \Delta Y$ the streamlines obey an equation

$$\tfrac{1}{2}\mu^2 Y^2 + \epsilon\mu^{\frac{1}{2}}[W(t,0)\,\mathrm{e}^{\mathrm{i}kx} + \mathrm{c.c.}] + \dots = \mathrm{const},$$

or, with (3.3),

$$Y^2 - \frac{\mu^{-\frac{3}{2}}}{(\pi k)^{\frac{1}{2}}}\cos(kx - \tfrac{1}{4}\pi)\int_C \mathrm{d}s\, s^{-\frac{3}{2}}A(t-s)\,\mathrm{d}s + \dots = \mathrm{const}. \tag{4.6}$$

Dots denote the contributions due to the zeroth, the second and other harmonics. As $A(t)$ oscillates for $\eta < 1$, the integral in (4.6) becomes equal to zero from time to time and 'cat's eyes' with the fundamental wavelength $\lambda = \lambda_0 = 2\pi/k$ vanish. For a short time they are replaced by a pattern with the wavelength $\lambda = \frac{1}{2}\lambda_0$ (as $\Psi_2^{(1)}(t, Y=0)$ is not zero at the same time) and then 'cat's eyes' with $\lambda = \lambda_0$ are regenerated but shifted by one half-period with respect to an initial position so that the 'pupil' of the new eye coincides with the corner of the old one. Far from the separatrix the half-period shift of a pattern takes place at other moments of time exactly when the sign of $A(t)$ changes.

Therefore the results of the numerical analysis demonstrate that for every Prandtl number η the nonlinearity sooner or later becomes dominant and the amplitude grows with no limitations (in a monotonic manner when $\eta > 1$ and with oscillation when $\eta < 1$).

Let us discuss now the frames of validity of (4.1). Strictly speaking, (4.1) is derived for the region $\gamma_{\mathrm{L}} > \nu^{\frac{1}{3}}$ where the CL regime is unsteady up to an infinitesimally small amplitude. That is why in this region of parameters we can formulate the initial conditions ($A(t) = A_0\,\mathrm{e}^{\gamma_{\mathrm{L}}t}, t \rightarrow -\infty$) which determine the unique solution. However, it is virtually obvious that (4.1) also describes the evolution of perturbations above the region of a viscous CL, i.e. when $0 < \gamma_{\mathrm{L}} \ll \nu^{\frac{1}{3}}$, $\epsilon \gg \nu^{\frac{2}{3}}$, but one must delete the linear term from the right-hand side because the threshold of nonlinearity is already exceeded. However, it is impossible in this case to formulate the initial condition in such a way as to determine the unique solution. So, we know that the solution is described in this case by (4.4), but we cannot find t_0 (it must be remembered that the perturbation reaches the upper boundary of a viscous CL region only when $\eta > 1$). To find t_0 one must derive an equation that is valid in both regimes of CL and solve it with the 'initial' condition $A(t) = A_0\,\mathrm{e}^{\gamma_{\mathrm{L}}t}$ as $t \rightarrow -\infty$. However, this task will not give essentially new information and hence is unjustifiably complicated.

The results of the solution of (4.1) with cubic nonlinearity have an intermediate character since this equation is only a particular case of the complete evolutionary equation (1.3) (see §1) and it is valid when $A < \nu^{\frac{1}{2}}$. The unbounded growth of amplitude at any sign of $\eta - 1$ found by us shows that sooner or later the amplitude will exceed this level and nonlinearity A^5 will come into play. That is why the 'burst-like' phase (4.4) when $\eta > 1$ and oscillations when $\eta < 1$ are only an intermediate asymptotic of solution.

4.2. *Quintic nonlinearity*

Equation (4.2) does not involve the coefficients κ and ν and may, therefore, be considered from two points of view, namely as being a full nonlinear equation in an idealized non-dissipative flow or as an equation describing an appropriate stage of development of a perturbation of an initially small amplitude as it exceeds the level $A \sim O(\nu^{\frac{1}{2}})$.

(i) The nonlinear term in (4.2), as in (4.1) and in Paper 1, is due totally to the interaction of harmonics inside the CL, hence (4.2) has a rather universal character and can readily be generalized to a broad class of solutions. The value of $(q^+ - q^-)$ here depends on the flow structure, except for the wavenumber k of the most unstable mode and the coefficient that relates the linear growth rate γ_L to supercriticality $\frac{1}{4} - J$. That value is zero for a flow with an antisymmetric velocity profile. (In (4.2) this dependence has been made apparent by incorporating the coefficient λ for a flow with $q^+ \neq q^-$, whereas for the antisymmetric profile considered here, $\lambda = 2^{\frac{9}{2}} k^7 \pi^{-\frac{5}{2}} \Gamma^{-2}(\frac{1}{4})$.

(ii) As in the case of cubic nonlinearity and for the same reason the equation is invariant under the shift in time.

(iii) Despite the still more complicated integral character of the nonlinear term in (4.2) as compared with (4.1), equation (4.2) – by virtue of the positive kernel $H(x, y, z)$ – is, in some sense, simpler than (4.1). Indeed, that $H(x, y, z)$ is positive means the montonic increase of the perturbations (with a real, positive λ). It is easy to reveal the asymptotic behaviour of the originally small perturbation.

At the initial stage, as long as the amplitude is very small ($A \ll \gamma_L^{\frac{7}{4}}$) nonlinearity gives only small corrections to the exponential growth $A = A_0 \, e^{\gamma_L t}$, and the evolution equation has the form

$$\frac{dA}{dt} = \gamma_L A + a_4 |A|^4 A, \tag{4.7}$$

where

$$a_4 = 2^{-5} 3^{-4} (2\pi)^{-\frac{1}{2}} \Gamma^{-2}(\tfrac{1}{4}) \left[5\mathbb{E}(\tfrac{1}{3}) - 6\mathbb{K}(\tfrac{1}{3}) + 6 \times 2^{\frac{1}{2}} \int_0^1 dx \frac{(1-x)^{-\frac{1}{2}} \arctan(x^{\frac{1}{2}})}{(1+x)^{\frac{3}{2}} (1+x/3)^{\frac{1}{2}}} \right] k^7 \gamma_L^{-6}$$
$$\approx 1.59 \times 10^{-5} k^7 \gamma_L^{-6} > 0,$$

$\mathbb{K}$ and $\mathbb{E}$ are full elliptical integrals of the first and second kinds. Positivity of a_4 indicates that nonlinearity accelerates the growth. When the amplitude passes through the nonlinearity threshold $A \sim O(\gamma_L^{\frac{7}{4}})$, the nonlinear term in (4.2) becomes dominant. It is a straightforward task to see that in this case the perturbation development acquires an explosive character and is represented by a power-law expression:

$$A = A_*(t_0 - t)^{-\frac{7}{4}}, \tag{4.8}$$

where

$$A_*^4 = \frac{7}{16} \left(\frac{\pi}{2} \right)^{\frac{5}{2}} \Gamma^2(\tfrac{1}{4}) \, k^{-7} I_5^{-1}$$

and I_5 is equal to

$$I_5 = \int_0^1 d\sigma \, \sigma^5 (1-\sigma)^{\frac{7}{4}} \int_0^1 dx \int_0^1 dy \int_0^1 dz \, H(x, y, z) [1 - \sigma(1-x)]^{-\frac{7}{4}} [1 - \sigma(1-xy)]^{-\frac{7}{4}}$$
$$\times [1 - \sigma(1 - 2xyz)]^{-\frac{7}{4}} [1 + \sigma x(1 + y - 2yz)]^{-\frac{7}{4}} \approx 5 \times 10^{-4}.$$

The above picture of the evolution is realized in all flows with an antisymmetric velocity profile (where the CL coincides with the symmetry plane) as well as in a broader class of flows for which $q^+ = q^-$ (see (2.7)). If, however, we have $q^+ \neq q^-$ (and such a possibility cannot, perhaps, be excluded from general considerations), then, as follows from (3.21), the coefficient appearing in (4.2) will be a complex one, $\arg \lambda = -\arctan[(q^+ - q^-)/\pi]$. The solution will be a rather complicated complex function; however, it seems to us that the qualitative character of the evolution can be established in this case as well (though with a lesser degree of confidence).

Equation (4.2) (without the first term on the right) also has a power-law solution

$$A = A_*(t_0 - t)^{-\frac{7}{4} + i\beta}. \tag{4.9}$$

Substitution in (4.2) yields the equality

$$|A_*|^4 = (\tfrac{7}{4}-\mathrm{i}\beta)\,\lambda^{-1}I_\beta^{-1},$$

where

$$I_\beta = \int_0^1 \mathrm{d}\sigma\,\sigma^5(1-\sigma)^{\frac{7}{4}-\mathrm{i}\beta}\int_0^1 \mathrm{d}x\int_0^1 \mathrm{d}y\int_0^1 \mathrm{d}z\,H(x,y,z)\,[1-\sigma(1-x)]^{-\frac{7}{4}+\mathrm{i}\beta}$$
$$\times[1-\sigma(1-xy)]^{-\frac{7}{4}+\mathrm{i}\beta}[1-\sigma(1-2xyz)]^{-\frac{7}{4}-\mathrm{i}\beta}[1+\sigma x(1+y-2yz)]^{-\frac{7}{4}-\mathrm{i}\beta},$$

which can occur only for β such that

$$\arg[(\tfrac{7}{4}+\mathrm{i}\beta)\,\lambda I_\beta] = 2\pi n, \quad n = 0, \pm 1, \pm 2, \ldots.$$

This condition serves for the determination of β.

The evolution of an originally small perturbation seems to pass through the following stages. In the intial stage $A(t)$ grows exponentially. The nonlinearity, on the one hand, contributes to this growth and, on the other, changes $\arg A$. The result may be visualized in the form of an untwisting spiral on a complex plane A; at sufficiently large $|A|$ this spiral is described by (4.9).

This has been merely a hypothetical scenario of the evolution. We are unaware of any flow models in which $q^+ \neq q^-$; therefore, we have not undertaken a more detailed (with the help numerical solution, for example) study of the perturbation evolution that is described by (4.2) with a complex λ.

5. Discussion

In this study we have derived for the regime of an unsteady CL the nonlinear equations (4.1) and (4.2) governing (each in its respective region of parameters, see figure 1) the time evolution of perturbations in a weakly supercritical shear flow of stratified fluid. Based on them one can develop a combined evolution equation that is valid throughout the entire region of an unsteady CL:

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \gamma_{\mathrm{L}}A+(\eta-1)\,\nu b_2\int_0^\infty \mathrm{d}s\,s^5\int_0^1 \mathrm{d}\sigma\,\sigma G(\sigma)\,A(t-s)\,A(t-\sigma s)\,\bar{A}(t-s-\sigma s)$$
$$+b_4\int_0^\infty \mathrm{d}s\,s^5\int_0^1 \mathrm{d}x\int_0^1 \mathrm{d}y\int_0^1 \mathrm{d}z\,H(x,y,z)\,A(t-s)\,A(t-xs)\,A(t-xys)$$
$$\times\bar{A}(t-2xyzs)\,\bar{A}(t-(1+x+xy-2xyz)\,s), \tag{5.1}$$

where $b_2 = k^6/12\pi\eta$, $b_4 = 2^{\frac{9}{2}}k^7/\pi^{\frac{5}{2}}\Gamma^2(\frac{1}{4})$ are positive constants.

As an 'initial' condition for its solution we have made the assumption that the perturbation evolves from the linear eigenfunction with a vanishingly small (at $t\rightarrow-\infty$) amplitude. This has saved us from specifying the 'traditional' initial condition, i.e. the spatial structure not only of the fundamental but also of all the other harmonics (at least those that were required in §3 for the derivation of the evolution equation) at $t = 0$.

In such a 'traditional' formulation the problem of non-dissipative flow was solved by Brown & Stewartson (1978). However, they were unsuccessful in surmounting the difficulty concerned. On the one hand, although the initial problem was solved, the initial conditions were not formulated explicitly and, on the other, their evolution equation has a rather limited range of validity.

We shall show that it is derivable from (5.1) if we assume $A = 0$ as $t < 0$ and consider only sufficiently small t, at which the amplitude can be taken equal to the initial (at $t = 0$) value. Indeed, at $\nu = \kappa = 0$ the cubic term in (5.1) vanishes and we have

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \gamma_{\mathrm{L}} A + \lambda \,|\, A \,|^4 A \int_0^t \mathrm{d}s\, s^5 \int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y \int_0^1 \mathrm{d}z H(x, y, z)\, \theta(t - 2sxyz)$$
$$\times\, \theta(t - (1 + x + xy - 2xyz)\, s) = \gamma_{\mathrm{L}} A + \lambda_1 k^7 t^6 |A|^4 A, \quad \lambda_1 \approx 6 \times 10^{-6},$$

which exactly coincides with equation (8.1) given in Brown & Stewartson (1978) (they only give the order of magnitude of λ_1). This equation is valid only for small t such that the amplitude has not yet been able to alter substantially and, therefore, describes only the trend of development rather than the development of initial perturbations.

This paper completes our study of the possible regimes of development of initially small perturbations in a weakly supercritical, stratified shear flow. The results obtained provide a complete picture of the evolution. Let us suppose that the initial perturbation amplitude is very small such that the perturbation begins right at the bottom of the diagram shown in figure 1. In the early stage, nonlinearity is non-competitive and the perturbation grows exponentially ($A \sim \mathrm{e}^{\gamma_{\mathrm{L}} t}$) with a linear growth rate γ_{L}. The subsequent fate of the perturbation depends on the super-criticality (i.e. on γ_{L}) as well as other parameters of the problem.

If the supercriticality is small enough ($\gamma_{\mathrm{L}} < \nu^{\frac{1}{3}}$), the critical layer is viscous initially and the evolution equation has the form (Paper 1)

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \gamma_{\mathrm{L}} A + \frac{b}{\nu}(\eta - 1)\, A^3. \tag{5.2}$$

With the Prandtl number $\eta < 1$, nonlinearity stabilizes the perturbation growth at the level $A = A_0 = O(\gamma_{\mathrm{L}}^{\frac{1}{2}} \nu^{\frac{1}{2}})$. When $\eta > 1$, nonlinearity, on the contrary, raises the rate of perturbation growth and renders the evolution 'explosive': $A \sim (t_0 - t)^{-\frac{1}{2}}$. Once the amplitude $A = O(\nu^{\frac{2}{3}})$ is attained, the CL regime changes to unsteady and the subsequent evolution is now governed by equation (5.1). Upon passing into the unsteady CL regime, the perturbation growth continues, with its 'explosive' character unaltered. When $\nu^{\frac{2}{3}} < A < \nu^{\frac{1}{2}}$ cubic nonlinearity is dominant and the amplitude increases as $A \sim (t_1 - t)^{-\frac{7}{2}}$; after that, nonlinearity A^5 becomes dominant and the amplitude varies now up to $A \sim O(1)$ according to (4.8), as $A \sim (t_2 - t)^{-\frac{7}{4}}$.

Thus, when $\gamma_{\mathrm{L}} < \nu^{\frac{1}{3}}$, the perturbation either is stabilized at a sufficiently low level (if $\eta < 1$) or grows infinitely (if $\eta > 1$), evolving through four stages, in each of which the time dependence of the amplitude is determined by an appropriate intermediate asymptotic:

$$A \sim \mathrm{e}^{\gamma_{\mathrm{L}} t}, \qquad A < A_0 \quad ; \qquad A \sim (t_0 - t)^{-\frac{1}{2}}, \quad A_0 < A < \nu^{\frac{2}{3}}; \tag{5.3a,b}$$
$$A \sim (t_1 - t)^{-\frac{7}{2}}, \quad \nu^{\frac{2}{3}} < A < \nu^{\frac{1}{2}} \; ; \qquad A \sim (t_2 - t)^{-\frac{7}{4}}, \quad \nu^{\frac{1}{2}} < A < 1. \tag{5.3c,d}$$

The subsequent evolution of the perturbation is associated with a severe restructuring of the flow and is not described by weakly nonlinear theory.

In the case of a higher supercriticality ($\nu^{\frac{1}{3}} < \gamma_{\mathrm{L}} < 1$), the perturbation evolves all the time in the unsteady CL regime, in accordance with (5.1). Initially, the amplitude grows exponentially and then nonlinear terms come into play. There is only a narrow region ($\nu^{\frac{1}{3}} < \gamma_{\mathrm{L}} < \nu^{\frac{2}{7}}$) in which the competition between nonlinearities may take place. When $\eta > 1$, this competition leads merely to the replacement of one 'explosive' regime ($A \sim (t_3 - t)^{-\frac{1}{2}}$) by another ($A \sim (t_4 - t)^{-\frac{1}{4}}$). When $\eta < 1$, a much more interesting picture might emerge. Cubic nonlinearity specifies, in the range

$\nu^{\frac{2}{3}} < A < \nu^{\frac{1}{2}}$, a rather peculiar regime of amplitude variation, viz. oscillations with an ever increasing swing. Ultimately, nonlinearity A^5 comes into play and then the perturbation grows monotonically according to (4.8): $A \sim (t_5 - t)^{-\frac{7}{4}}$. However, it seems difficult to realize such a regime in practice because even at quite a large Reynolds number (10^6, for example, i.e. $\nu = 10^{-6}$) cubic nonlinearity is dominant in a relatively small range of amplitudes: $10^{-4} \ll A \ll 10^{-3}$, so that the relevant intermediate asymptotic may not have enough time to become established. Finally, as $\gamma_L > \nu^{\frac{2}{7}}$, cubic nonlinearity becomes non-competitive – either compared with the linear term or compared with nonlinearity A^5 – and has virtually no influence upon the development of the perturbation, which increases monotonically, first exponentially and then 'explosively' according to (4.8).

Three points should be emphasized in conclusion. First, it appears that, as the initially small perturbation evolves, the nonlinear CL regime cannot be realized: the growth rate of the perturbations with increasing amplitude increases so rapidly that the width of the unsteady CL ($l_t \sim \gamma$) is at all times greater than the nonlinear CL width ($l_N \sim A^{\frac{2}{3}}$). (This does not mean, however, that the existence of a nonlinear CL in a stratified flow is always impossible. It can appear in, for example, the problem of the interaction between internal gravity waves of a finite amplitude and a shear flow).

Second, the evolution equation (5.1) is a fully sufficient one up to amplitudes $A \sim O(1)$ in the sense that it is not necessary to take higher-order nonlinear terms into account. The reason for this has already been discussed: that is the expansion of the nonlinear term in the evolution equation proceeds in the parameter A^2/l^3 (where $l \sim l_t \sim \gamma$ is the width of the unsteady CL), which remains small during the course of the evolution: $A^2/l^3 = O(A^{\frac{2}{7}}) \ll 1$.

And third, nonlinear terms of (5.1) do not depend on the flow structure as a whole; therefore, (5.1) is readily extended to virtually any weakly supercritical, stratified shear flow with a monotonic velocity profile. Besides, (5.1) is easy to generalize to the case of perturbations modulated in x. A more detailed discussion of these questions has been given in Paper 1.

We are grateful to Professor V. D. Shapiro for his interest. Thanks are also due to Mr V. G. Mikhalkovsky for his assistance in preparing the English version of the manuscript and for typing and retyping the text.

Appendix A. The solution of the equation $\hat{\mathrm{N}}F = R$

For the lth harmonic

$$\hat{\mathrm{N}} \equiv \hat{\mathrm{N}}_l = \left(\frac{\partial}{\partial \tau} + \mathrm{i}lkY\right)\mathscr{D} - \tfrac{1}{2}\mathrm{i}lk.$$

As $l \neq 0$, then

$$\mathrm{N}_l = \mathrm{i}lk\left[\left(Y - \frac{\mathrm{i}}{lk}\frac{\partial}{\partial \tau}\right)\mathscr{D} - \tfrac{1}{2}\right]$$

and all essential properties of the solution can be demonstrated by taking the fundamental ($l = 1$) as an example. The zeroth harmonic does not keep within this scheme and all related calculations are made in the main text. In order to be as close as possible to the problem solved in this paper, let us consider the system of equations

$$\hat{\mathrm{N}}_1 F = \tfrac{1}{2}\mathrm{i}kG + R_1, \quad \hat{\mathrm{N}}_1 G = R_2 \tag{A 1}$$

which is equivalent to

$$\hat{\mathrm{N}}_1^2 F = R, \quad R = \hat{\mathrm{N}}_1 R_1 + \tfrac{1}{2}\mathrm{i}kR_2. \tag{A 2}$$

A.1. The homogeneous equations $\hat{\mathrm{N}}_1 W = 0$ *and* $\hat{\mathrm{N}}_1^2 U = 0$

Let us define the Fourier transformation as usual:

$$X = \int_{-\infty}^{\infty} x(\omega)\,\mathrm{e}^{\mathrm{i}\omega\tau}\,\mathrm{d}\omega, \quad x(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\tau)\,\mathrm{e}^{-\mathrm{i}\omega\tau}\,\mathrm{d}\tau.$$

The equations under consideration in terms of the Fourier components take the form

$$\hat{\mathrm{N}}_{1\omega}\, w \equiv \mathrm{i}k(z\mathscr{D} - \tfrac{1}{2})\, w = 0, \quad \hat{\mathrm{N}}_{1\omega}^2\, u \equiv -k^2(z^2\mathscr{D}^2 + \tfrac{1}{4})\, u = 0; \quad z = Y + \frac{\omega}{k}.$$

Their solutions are

$$w = b(\omega) z^{\frac{1}{2}}, \quad u = z^{\frac{1}{2}}[c(\omega) + d(\omega)\ln(\tfrac{1}{2}\mu z)].$$

Let us calculate W. For this purpose it is convenient first to calculate W_Y:

$$W_Y = \frac{1}{2}\int_{-\infty}^{\infty} \mathrm{d}\omega\,\mathrm{e}^{\mathrm{i}\omega\tau}\, b(\omega)\, z^{-\frac{1}{2}} = \frac{1}{4\pi}\int_{-\infty}^{\infty}\mathrm{d}t\, B(t)\int_{-\infty}^{\infty}\mathrm{d}\omega\left(Y + \frac{\omega}{k}\right)^{-\frac{1}{2}}\mathrm{e}^{\mathrm{i}\omega(\tau - t)}.$$

According to the causality principle, the value of W_Y at time τ must depend on the past ($t < \tau$) only: the future cannot contribute to it. Permitting ω to be complex-valued we see that the causality requires the integrand to be analytic in the lower half-plane, i.e. as $\operatorname{Im}\omega < 0$. A simple rearrangement provides

$$W_Y = \tfrac{1}{2}\,\mathrm{e}^{\mathrm{i}\pi/4}\left(\frac{k}{\pi}\right)^{\frac{1}{2}}\int_0^{\infty}\mathrm{d}t\, t^{-\frac{1}{2}} B(\tau - t)\,\mathrm{e}^{-\mathrm{i}kYt}.$$

If this is integrated in Y directly, we obtain a divergent integral in it. Therefore we first extend the integration in t to contour C in the complex plane dissected by the ray $\arg t = -2\pi$ (figure 2*a*):

$$W_Y = \tfrac{1}{4}\,\mathrm{e}^{\mathrm{i}\pi/4}\left(\frac{k}{\pi}\right)^{\frac{1}{2}}\int_C \mathrm{d}t\, t^{-\frac{1}{2}} B(\tau - t)\,\mathrm{e}^{-\mathrm{i}kYt}$$

and then integrate in Y:

$$W = \frac{\mathrm{e}^{3\mathrm{i}\pi/4}}{4(\pi k)^{\frac{1}{2}}}\int_C \mathrm{d}t\, t^{-\frac{3}{2}} B(\tau - t)\,\mathrm{e}^{-\mathrm{i}kYt}. \tag{A 3}$$

In a similar way

$$U = \frac{\mathrm{e}^{3\mathrm{i}\pi/4}}{4(\pi k)^{\frac{1}{2}}}\int_C \mathrm{d}t\, t^{-\frac{3}{2}}\,\mathrm{e}^{-\mathrm{i}kYt}\left\{C(\tau - t) + D(\tau - t)\left[\psi(\tfrac{3}{2}) - \ln\frac{2kt}{\mu} - \frac{3\mathrm{i}\pi}{2}\right]\right\}. \tag{A 4}$$

The functions W and U are analytic in the lower half-plane of complex-valued Y and tend to zero as $\tau \to -\infty$, if $B(\tau)$, $C(\tau)$ and $D(\tau)$ tend to zero. From a solution in the case of a non-competitive nonlinearity we know that they tend to zero exponentially ($\sim \mathrm{e}^{\Gamma\tau}$) and in view of the theory of Laplace transformation this means the analyticity of W and U in a more extended domain, namely in the half-plane $\operatorname{Im} Y < \Gamma/k$. As $|Y| \to \infty(-\pi \leqslant \arg Y \leqslant 0)$ the functions W and U have the following asymptotic representations:

$$\left.\begin{aligned} W &= B(\tau)\, Y^{\frac{1}{2}} - \frac{\mathrm{i}}{2k}\frac{\partial B}{\partial \tau} Y^{-\frac{1}{2}} + O(Y^{-\frac{3}{2}}), \\ U &= C(\tau)\, Y^{\frac{1}{2}} + D(\tau)\, Y^{\frac{1}{2}}\ln(\tfrac{1}{2}\mu Y) + O(Y^{-\frac{1}{2}}\ln(\tfrac{1}{2}\mu Y)). \end{aligned}\right\} \tag{A 5}$$

Note that the equation $\hat{N}_1^2 U = 0$ is equivalent to $\hat{N}_1 U = ikW_D$, where W_D is obtained from (A 3) after substituting $D(\tau)$ instead of $B(\tau)$.

A.2. *The equation* $N_1 F = R$

In terms of Fourier components we have

$$zf_Y - \tfrac{1}{2}f = -\frac{i}{k}r, \quad f = -\frac{i}{k}z^{\frac{1}{2}}\int_{-\infty}^{Y} ds\, z^{-\frac{3}{2}}r(s). \tag{A 6}$$

After the inverse transformation we obtain in the same manner as in the preceding Subsection

$$\begin{aligned} F &= -\frac{1}{2\pi}\int_0^\infty ds \int_0^\infty dt\, R(\tau-t, Y-s)\, e^{-ikYt}\int_L du\, e^{ikstu}u^{\frac{1}{2}}(1-u)^{-\frac{3}{2}} \\ &\equiv \int_0^\infty ds \int_0^\infty dt\, R(\tau-t, Y-s)\, e^{-ikYt}\Phi(\tfrac{3}{2}, 1; ikst). \end{aligned} \tag{A 7}$$

Here $\Phi(a, c; z)$ is a Kummer's confluent hypergeometric function (Erdélyi 1953) and contour L for integrating in u is shown in figure 2(*b*).

In this Subsection we have obtained a particular solution of the equation. To obtain the general solution one needs to add an arbitrary solution of a homogeneous equation (W). Solving (A 1) is reduced to the consistent integration of two equations of the type just considered. The results for the lth harmonic are obtained by replacement of k by lk.

A.3. *The asymptotic expansions*

If $R \to 0$ as $Y \to \pm\infty$, then in (A 6) we have

$$f = -\frac{i}{k}Y^{\frac{1}{2}}\int_{-\infty}^{\infty} ds\, z^{-\frac{3}{2}}r(s) + O(Y^{-\frac{1}{2}}),$$

whence

$$\begin{aligned} F &\sim -\frac{i}{2\pi k}Y^{\frac{1}{2}}k^{\frac{3}{2}}\int_{-\infty}^{\infty} ds \int_0^\infty dt\, R(\tau-t, s)\, e^{-ikst}\int_{-\infty-i0}^{\infty-i0} du\, u^{-\frac{3}{2}}\, e^{iut} \\ &= 2\left(\frac{k}{\pi}\right)^{\frac{1}{2}} e^{i\pi/4}Y^{\frac{1}{2}}\int_{-\infty}^{\infty} ds \int_0^\infty dt\, t^{\frac{1}{2}}R(\tau-t, s)\, e^{-ikst}, \end{aligned}$$

so that

$$F = 2\left(\frac{k}{\pi}\right)^{\frac{1}{2}} e^{i\pi/4}Y^{\frac{1}{2}}\int_{-\infty}^{\infty} ds \int_0^\infty dt\, t^{\frac{1}{2}}R(\tau-t, s)\, e^{-ikst} + O(Y^{-\frac{1}{2}}), \quad Y \to \infty. \tag{A 8}$$

The same result may be obtained from (A 7) if an asymptotic representation for Φ is used (as R is localized, the main contribution to the integral at large Y is due to $Y - s = O(1)$), i.e. to $|s| \gg 1$). We have obtained an asymptotic representation of the *particular* solution as $Y \to +\infty$ (as $Y \to -\infty$, $F = O(Y^{-\frac{1}{2}})$). For a general solution we have evidently the representation

$$F = C^{\pm}Y^{\frac{1}{2}} + O(Y^{-\frac{1}{2}}), \quad C^+ - C^- = 2\left(\frac{k}{\pi}\right)^{\frac{1}{2}} e^{i\pi/4}\int_{-\infty}^{\infty} dY \int_0^\infty dt\, t^{\frac{1}{2}}R(\tau-t, Y)\, e^{-ikYt}. \tag{A 9}$$

Now we consider the question of finding an asymptotic expansion for the solution of (A 1) (or of (A 2) equivalent to it). In terms of Fourier components

$$\hat{N}_{1\omega}^2 f = -k^2(z^2\mathscr{D}^2 + \tfrac{1}{4})f = \hat{N}_{1\omega}r_1 + \tfrac{1}{2}ikr_2 \tag{A 10}$$

we can see that if r_1 and r_2 are localized in Y, then as $Y \to \pm\infty$

$$f = c^{\pm} Y^{\frac{1}{2}} + d^{\pm} Y^{\frac{1}{2}} \ln(\tfrac{1}{2}\mu Y) + O(Y^{-\frac{1}{2}} \ln(\tfrac{1}{2}\mu Y)). \tag{A 11}$$

To derive the evolutionary equation we have first to calculate $d^{\pm}$. Multiplying (A 10) by the eigenfunction v of the operator $\hat{\mathrm{N}}^{+}_{1\omega}$ conjugate to $\hat{\mathrm{N}}_{1\omega}$:

$$\hat{\mathrm{N}}^{+}_{1\omega} v \equiv -\mathrm{i}k(\mathscr{D}z + \tfrac{1}{2})v = -\mathrm{i}k(z\mathscr{D} + \tfrac{3}{2})v = 0, \quad v = z^{-\frac{3}{2}}$$

and integrating in Y we obtain

$$\int_{-\infty}^{\infty} \mathrm{d}Y\, v\hat{\mathrm{N}}^{2}_{1\omega} f = \int_{-\infty}^{\infty} \mathrm{d}Y\, v(\hat{\mathrm{N}}_{1\omega} r_1 + \tfrac{1}{2}\mathrm{i}k\, r_2) = \int_{-\infty}^{\infty} \mathrm{d}Y(r_1 \hat{\mathrm{N}}^{+}_{1\omega} v + \tfrac{1}{2}ik\, vr_2) = \tfrac{1}{2}ik \int_{-\infty}^{\infty} \mathrm{d}Y\, vr_2.$$

Using (A 11) we see that the first integral is

$$\int_{-\infty}^{\infty} \mathrm{d}Y\, v\hat{\mathrm{N}}^{2}_{1\omega} f = -k^2(vz^2 f' - (vz^2)' f)\Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \mathrm{d}Y f(\hat{\mathrm{N}}^{+}_{1\omega})^2 v = -k^2(d^+ - d^-).$$

Therefore

$$d^+ - d^- = -\frac{\mathrm{i}}{2k} \int_{-\infty}^{\infty} \mathrm{d}Y z^{-\frac{3}{2}} r_2. \tag{A 12}$$

After the inverse Fourier transformation we see that as $Y \to \pm\infty$

$$F = C^{\pm} Y^{\frac{1}{2}} + D^{\pm} Y^{\frac{1}{2}} \ln(\tfrac{1}{2}\mu Y) + O(Y^{-\frac{1}{2}} \ln(\tfrac{1}{2}\mu Y)), \tag{A 13}$$

where

$$C^{\pm} = \int_{-\infty}^{\infty} \mathrm{d}\omega\, \mathrm{e}^{\mathrm{i}\omega\tau} c^{\pm}(\omega), \quad D^{\pm} = \int_{-\infty}^{\infty} \mathrm{d}\omega\, \mathrm{e}^{\mathrm{i}\omega\tau} d^{\pm}(\omega).$$

In particular

$$D^+ - D^- = \left(\frac{k}{\pi}\right)^{\frac{1}{2}} \mathrm{e}^{\mathrm{i}\pi/4} \int_{-\infty}^{\infty} \mathrm{d}Y \int_{0}^{\infty} \mathrm{d}t\, t^{\frac{1}{2}} R_2(\tau - t, Y)\, \mathrm{e}^{-\mathrm{i}kYt} \tag{A 14}$$

Finally, as R is not localized (i.e. does not tend to zero as $Y \to \pm\infty$) the calculation of an asymptotic expansion requires a special consideration. For example, one can separate the non-localized part in R and try to obtain for it the exact solution of (A 7) type and then calculate an asymptotic expansion.

REFERENCES

ABRAMOWITZ, M. & STEGUN, I. A. 1964 *Handbook of Mathematical Functions*. Natl. Bureau of Standards.

BROWN, S. N., ROSEN, A. S. & MASLOWE, S. A. 1981 The evolution of a quasi-steady layer in a stratified viscous shear layer. *Proc. R. Soc. Lond.* A **375**, 271.

BROWN, S. N. & STEWARTSON, K. 1978 The evolution of a small inviscid disturbance to a marginally unstable stratified shear flow: stage two. *Proc. R. Soc. Lond.* A **363**, 175.

CHURILOV, S. M. & SHUKHMAN, I. G. 1987*a* Nonlinear stability of a stratified shear flow: viscous critical layer. *J. Fluid Mech.* **180**, 1 (Paper 1).

CHURILOV, S. M. & SHUKHMAN, I. G. 1987*b* Nonlinear development of disturbances in a zonal shear flow. *Geophys. Astrophys. Fluid Dyn.* **38**, 145.

DRAZIN, P. G. 1958 The stability of a shear layer in an unbounded heterogeneous inviscid fluid. *J. Fluid Mech.* **4**, 214.

ERDÉLYI, A. 1953 *Higher Transcendental Functions*, vol 1. McGraw-Hill.

HICKERNELL, F. J. 1984 Time-dependent critical layers in shear flows on beta-plane. *J. Fluid Mech.* **142**, 431.

LANDAU, L. D. & LIFSHITZ, E. M. 1959 *Fluid Mechanics*. Pergamon.